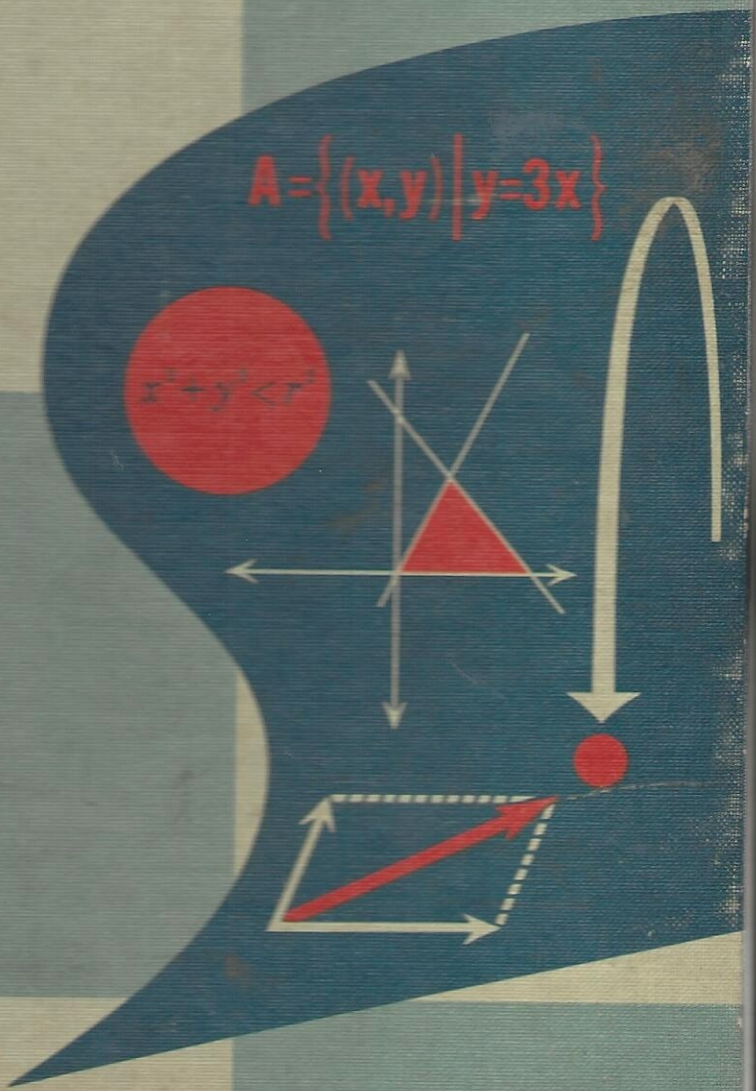
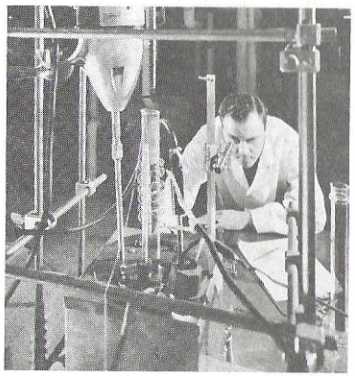
MODERN MATHEMATICS



ALGEBRA ONE

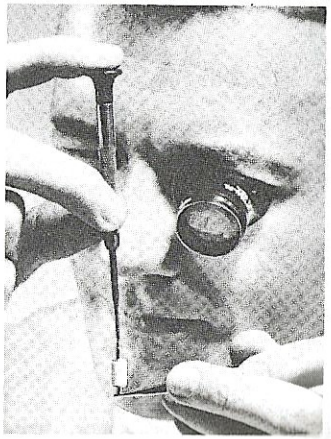
PART 1 OF 3

MODERN MATHEMATICS *Algebra One*



Thus,

$$1+4+7+\cdots+(3k-2)+(3k+1)$$
$$=\tfrac{1}{2}k(3k-1)+(3k+1)$$
$$=\tfrac{3}{2}k^2-\tfrac{1}{2}k+3k+1$$
$$=\tfrac{3}{2}k^2+\tfrac{5}{2}k+1$$
$$=\tfrac{1}{2}(3k^2+5k+2)$$
$$=\tfrac{1}{2}(k+1)(3k+2)$$



MODERN MATHEMATICS

Algebra One

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Contents

1. An Introduction to Algebra 1

Mathematical Systems and Their Properties
A Lever System • Defining the ⊕ Operation • Closure • Modulo 7 and Modulo 3 • A System without Numbers • A Generalized System • Properties of a System

Sets and Sentences
Representing Sets • Inequality Symbols • Replacement Sets • Open Sentences and Statements • Set Notation • Graph of a Set

Numbers and Number Relationships
Numerals • Exponents • Expressions • Multiplication and Exponents • Division and Exponents • Zero Exponent • Principles • Proving Number Relationships

2. The Integers and the Rational Numbers 42

A Further Look at Systems
Identity Element • Inverse Elements • Inverse Operations

The Integers
Constructing a New Set of Numbers • Interpreting Positive and Negative Numbers • Addition • Order • Opposites and Absolute Value • Subtraction • Commutative Group • Multiplication of Integers • Division • Uniqueness • More About Zero

The Rational Numbers
Identity Element and Inverse Elements for Multiplication • Division • Commutative Group under Multiplication • Addition of Rational Numbers • Subtraction • Commutative Group under Addition • Order • Betweenness • Density

3. Equations and Inequalities 88

Solving Equations and Inequalities
Equivalent Open Sentences • Solving Equations • Solving Inequalities • Problem Solving

Like Terms in Open Sentences

Problem Solving
Using Equations and Inequalities • Number Problems • Coin and Mixture Problems

Graphs of Solution Sets
Equations and Inequalities • Conjunction • Disjunction

4. Polynomials 122

Properties
Evaluating Polynomials • Principles of Operation • Using Polynomials in Problem Solving

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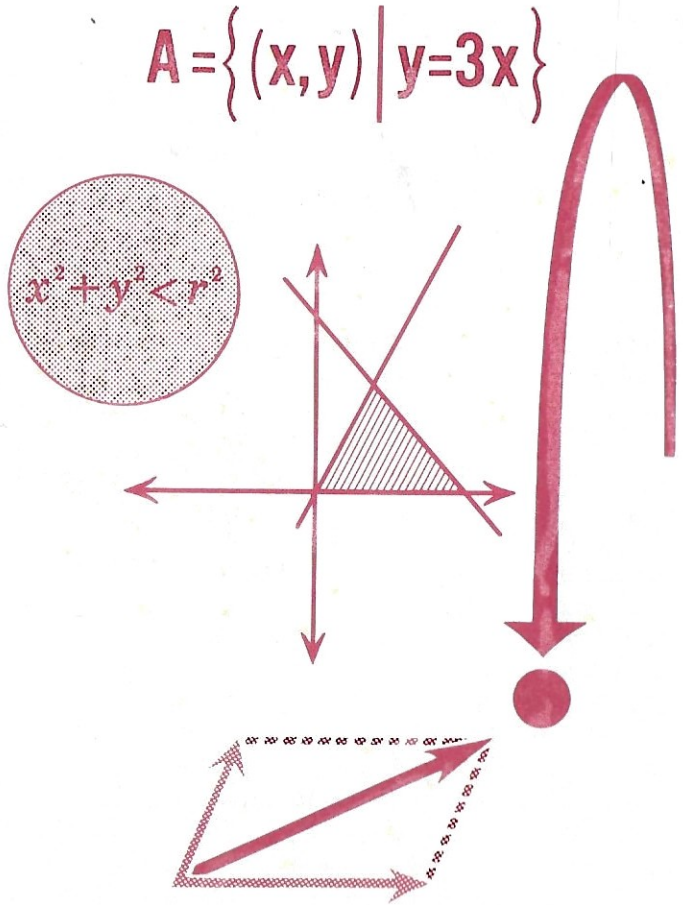
Addition and Subtraction
Addition • The Identity Polynomial and the Additive Inverse • Polynomials as a Commutative Group • Subtraction

Multiplying and Factoring Polynomials
Common Monomial Factors • Problem Solving • Multiplying Binomials • Multiplying a Polynomial by a Polynomial • Squaring a Binomial • Factoring Trinomial Squares • Product of the Sum and Difference of Two Terms • Factoring the Difference between Two Squares • Factoring Trinomials

Polynomials in More than One Variable



1 An Introduction to Algebra

MATHEMATICAL SYSTEMS AND THEIR PROPERTIES

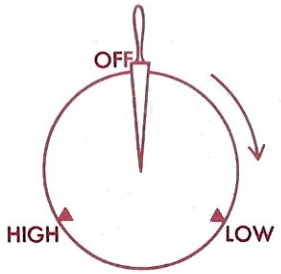
YOU ARE BEGINNING a year's work with a subject called "algebra." The roots of this subject extend far back into man's history and spread around the world.

Some of the topics of algebra you will study had their origins among scholars of ancient civilizations, particularly those of India and Greece. Other topics originated between 600 A.D. and 1450 A.D. among scholars of the Moslem civilization. A large part of what you will study was developed during the Renaissance, when Europeans improved upon the work of the earlier mathematicians and gave algebra much of its present form. Since the Renaissance, further advances have taken place, many of which are discussed in the chapters that follow. Today mathematicians throughout the world continue to invent new forms of algebra.

Nowhere in this book is *algebra* defined. The word is used to describe a great variety of mathematics, and no simple definition is possible. Instead of a definition, the year's work will begin with an exploration of some of the fundamental ideas of algebra. What do we mean by a *system* in mathematics? What are the properties of a system? How can we investigate number relationships and arrive at general statements of properties of numbers? These are only a few of the questions we will consider. From this investigation you will begin to appreciate what mathematicians have in mind when they speak of algebra.

A Lever System

In every area of mathematics, imagination can help you immensely. So let us begin with an imaginary situation.



The diagram at the left represents the control lever of a certain machine. The lever can be moved only in a clockwise direction, as indicated by the arrow. If the lever is at *Off*, the machine does not operate; if the lever is at *Low*, the machine moves at a low speed; if the lever is at *High*, the machine moves at a high speed. The lever cannot be stopped at any other points.

The design of the machine permits three choices in moving the lever. These choices we represent as X, Y, and Z as follows:

X: Not moving the lever (leaving it at *Off*, or leaving it at *Low*, or leaving it at *High*)

Y: Moving the lever clockwise to the next position (for example, from *Off* to *Low*, or from *Low* to *High*, or from *High* to *Off*)

Z: Moving the lever clockwise *through* the next position to the following position (for example, from *Off* to *High*, or from *Low* to *Off*, or from *High* to *Low*)

1. If we begin at *Off* and make a move of Y followed by another move of Y, where will the lever be? Would we have obtained the same result if we had begun at *Off* and made a move of Z? Since a move of Y followed by Y and a move of Z can be regarded as two names for the same move, we write: Y followed by $Y = Z$.

2. If we begin at *Off* and make a move of Y followed by a move of Z, where will the lever be? Would we have achieved the same result if we had left the lever at *Off*? We write: Y followed by $Z = X$. For simplicity's sake, let us use the symbol $\oplus$ to mean "followed by." Then we can write: $Y \oplus Y = Z$; $Y \oplus Z = X$.

3. What single move is equal to $X \oplus Y$? Begin at *Off*, make a move of X, then follow it with a move of Y. Would the result be the same if we made a move of Y? Does $X \oplus Y = Y$?

4. For each of the following pairs of moves, find the single move that gives the same result. Then write a true sentence such as $Y \oplus Y = Z$ to describe the relationship.

a. $Y \oplus X$ **b.** $Z \oplus Y$ **c.** $Z \oplus Z$
d. $X \oplus X$ **e.** $X \oplus Z$ **f.** $Z \oplus X$

To a mathematician, the lever and the ways in which it can be moved constitute a **system.** The lever system consists of three **elements** and an **operation.** The elements of the system are the moves represented by X, Y, and Z. The name of the operation of the system is "followed by," represented by the symbol $\oplus$.

5. Does $Y \oplus X$ give the same result as $X \oplus Y$? Then this sentence is true: $Y \oplus X = X \oplus Y$. Are the sentences below true?

a. $Y \oplus Z = Z \oplus Y$ **b.** $X \oplus Z = Z \oplus X$

6. In each of the sentences **a** and **b** above, do the same two elements appear at the left of the equal sign as at the right? Does the order in which the operation $\oplus$ is performed on a pair of elements appear to affect the result?

We say that the operation of $\oplus$ in the lever system is **commutative**; that is, the order in which we apply the operation to a pair of elements does not affect the result.

Defining the $\oplus$ Operation

In analyzing the properties of a system, a table can be helpful. Let us prepare a table that shows all the results that can be obtained in the lever system by applying the operation $\oplus$ to all pairs of elements of the system. To aid you, part of the table is shown at the right.

$\oplus$	X	Y	Z
X		Y	
Y			
Z		X	

Notice that the three elements of the system appear at the left and at the top of the table. The result of the operation $X \oplus Y$ appears opposite the X at the left and directly below the Y at the top: $X \oplus Y = Y$. Similarly, the result of $Z \oplus Y$ appears opposite the Z at the left and below the Y at the top: $Z \oplus Y = X$.

1. Copy the table and complete it. Refer to the diagram on page 2 if you need help.

2. Your complete table shows every possible result of the operation $\oplus$ with any two elements of the lever system. In the language of algebra, we say that the table **defines** the operation $\oplus$ for elements X, Y, and Z of the lever system.

3. The operation $\oplus$ is defined only for a pair of elements. Suppose, however, you are asked what $(Y \oplus Z) \oplus Z$ equals. You will recall from your earlier courses in mathematics that parentheses are a symbol of grouping, and that operations within parentheses are to be performed first.

Using the table, find what $Y \oplus Z$ equals. $Y \oplus Z = \underline{?}$. Next find $X \oplus Z$ in the table. What, then, does $(Y \oplus Z) \oplus Z$ equal?

The model example at the right shows how you would write the steps in the solution.

$$\begin{aligned}(Y \oplus Z) \oplus Z &= X \oplus Z \\ &= Z\end{aligned}$$

4. Find the result of $Y \oplus (Z \oplus Z)$ by first performing the operation within parentheses and then completing the other operation. Compare this result with the result for the example in **3.** Do you agree that the following sentence is true: $(Y \oplus Z) \oplus Z = Y \oplus (Z \oplus Z)$?

Using the table and the procedure shown above, find the results for the following. Remember in each case to perform the operation in parentheses first.

	a	b
5.	$(X \oplus Y) \oplus Z$	$X \oplus (Y \oplus Z)$
6.	$(Y \oplus Y) \oplus Z$	$Y \oplus (Y \oplus Z)$
7.	$(Z \oplus X) \oplus Y$	$Z \oplus (X \oplus Y)$

Compare **5a** and **b**; **6a** and **b**; **7a** and **b**. Are the following sentences true: $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$? $(Y \oplus Y) \oplus Z = Y \oplus (Y \oplus Z)$? $(Z \oplus X) \oplus Y = Z \oplus (X \oplus Y)$? In each of the sentences, do the same elements appear at the left of the equal sign as at the right? In what way do the operations on the left differ from the operations on the right?

We say that the operation $\oplus$ is **associative**; that is, where there are three elements to be combined by repeating the operation, we can perform the operation on any two adjacent elements first.

Comparing Two Systems

To provide a basis for comparing another system to the lever system, let us summarize some of the important ideas involved in the exploration of the lever system.

(1) How many elements are in the lever system?
(2) What is the operation in the system?
(3) Is the operation commutative?
(4) Is the operation associative?
(5) How many ways can we pair elements? (Check the table you made for **1**, page 3.)
(6) When we apply the operation $\oplus$ to a pair of elements, do we always obtain an element of the system as the result? (Check the table.)

1. Consider now a set of numbers consisting of all the numbers used in arithmetic—numbers such as 0, $\frac{1}{2}$, .7, 1, $1\frac{1}{4}$, 111.05, and so on. For convenience, let us call the set of all such numbers the **numbers of arithmetic.** Each of the six numbers mentioned above is then called an element of this set. Are these six numbers the only elements in the set of numbers of arithmetic? Are there more than twenty elements? more than a hundred? Do you think there are an unlimited number of elements in the set of numbers of arithmetic?

One operation that may be performed on the numbers of arithmetic is addition. Another operation is multiplication. Do you see that we can speak of a system involving the numbers of arithmetic?

2. Consider the operation of addition. Is it true that $5.4 + 4.6 = 4.6 + 5.4$? $1\frac{1}{2} + 1\frac{1}{4} = 1\frac{1}{4} + 1\frac{1}{2}$? $212.76 + 3.04 = 3.04 + 212.76$? Do you think that addition is commutative in the set of numbers of arithmetic?

3. Which of the following sentences are true?

a. $(212 + 36) + 72 = 212 + (36 + 72)$

b. $(3.7 + 4.6) + 2.8 = 3.7 + (4.6 + 2.8)$

c. $7\frac{1}{2} + (6\frac{3}{4} + 2\frac{1}{8}) = (7\frac{1}{2} + 6\frac{3}{4}) + 2\frac{1}{8}$

Did you perform the operations before deciding whether each sentence was true, or were you able to decide by inspection? If you decided by inspection, explain your reasoning. Do you agree that addition is associative in the set of numbers of arithmetic?

4. Since the lever system has only a limited, or *finite*, number of elements, we were able to make a table that showed us the results of all possible pairings of elements under the operation $\oplus$. Can we do this for addition of the numbers of arithmetic? Explain.

The table for the operation $\oplus$ in the lever system showed that the operation is commutative. We might even have made a table that would show all possible groupings involving three elements. We cannot do this for addition in the set of numbers of arithmetic. However, our experience leads us to *assume* that addition is commutative and associative.

Instead of making a list of specific statements about the numbers of arithmetic, we make a general statement. We say:

For every replacement for a and b by numbers of arithmetic,

$$a + b = b + a$$

We call this a **generalization.** It is a general statement of the commutative principle for addition of numbers of arithmetic.

If you could find one pair of replacements that makes the sentence false, we would say that addition in the set of numbers of arithmetic is not commutative.

5. Using a, b, and c to represent elements in the set of numbers of arithmetic, write a general statement of the associative principle for addition.

6. Consider the operation of multiplication in the set of numbers of arithmetic. Do you think it is commutative? associative? Write generalizations for the commutative and associative principles as they relate to multiplication in the set of numbers of arithmetic.

7. Examine the following sentences. Tell whether each is true.

a. $2 \div 2 = 2 \div 2$ **b.** $9 \div 3 = 3 \div 9$ **c.** $6 \div 4 = 4 \div 6$

Study the results of your work. In the set of numbers of arithmetic, is division commutative?

Even if you could find many examples in which $a \div b = b \div a$, it would still be false to say that division is commutative. Why?

8. Is it true that $(1 \div 1) \div 1 = 1 \div (1 \div 1)$? $(60 \div 6) \div 2 = 60 \div (6 \div 2)$? In the set of numbers of arithmetic, do you think division is associative?

Closure

Question (6) about the lever system (page 5) involves an extremely important property in mathematics. The question was: When we apply the operation $\oplus$ to a pair of elements, do we always obtain an element of the system as the result? The answer, you found, was: Yes; the result of applying the operation was always one of the elements, X, Y, or Z. The property that this illustrates is known as **closure**, and it may be described as follows:

When an operation on any two elements of a system yields a result that is also an element of the system, we say that the system is **closed** under that operation. We also say that the system has the property of closure for that operation.

1. To arrive at the answer for the lever system, you examined the entries in your table and found that there were nine possible ways we could pair elements under the operation $\oplus$, and that in each case we obtained as a result one of the elements in the system. We cannot, of course, examine all possible pairings of numbers of arithmetic under the operation of addition. But do you think that the numbers of arithmetic are *closed* under the operation of addition? That is, do you think the sum of any two numbers of arithmetic is always a number of arithmetic? Test your conclusion with a few examples.

2. Do you think the product of any two numbers of arithmetic is always a number of arithmetic? Try a few examples, such as $1\frac{1}{2} \times 5\frac{1}{2}$, to see if your judgment is reasonable.

If you agree that the product is always a number of arithmetic, then you can say: The numbers of arithmetic are *closed* under the operation of multiplication.

EXERCISES

In each of sentences **1–9**, find the number of arithmetic that you can use as a replacement for a, b, c, and so on, to make the sentence true. Then state what principle the sentence illustrates. If the principle is the: commutative principle of addition, write *CA*; associative principle of addition, write *AA*; commutative principle of multiplication, write *CM*; associative principle of multiplication, write *AM*.

1. $3 \times 7\frac{1}{2} = 7\frac{1}{2} \times a$ **2.** $(6.1 + 2.4) + c = 6.1 + (2.4 + 7.6)$

3. $2\frac{1}{4} + 18\frac{3}{4} = b + 2\frac{1}{4}$
4. $x \times (55 \times 73) = (48 \times 55) \times 73$
5. $(3\frac{1}{2} + 2\frac{1}{3}) + 1\frac{1}{4} = 3\frac{1}{2} + (y + 1\frac{1}{4})$
6. $456 \times 654 = 654 \times z$
7. $r + 65.53 = 65.53 + 33.47$
8. $4\frac{3}{8} \times (7\frac{2}{3} \times 8\frac{4}{5}) = (4\frac{3}{8} \times 7\frac{2}{3}) \times s$
9. $23.5 \times t = 15.6 \times 23.5$

10. Do you think it is true that the product of $7\frac{1}{103}$ and $3\frac{3}{89}$ is a number of arithmetic? What principle does this illustrate? Do you think it is also true that $(7\frac{1}{103} \times 3\frac{3}{89}) \times 6\frac{4}{5}$ would yield a number of arithmetic as the result? Explain your answer.

Two Familiar Sets of Numbers

1. Suppose you were asked to count the number of students in your algebra class. With what number would you begin the process of counting? What would be the next number? the next? Because the numbers we use for counting are among the numbers of arithmetic, it is sometimes difficult for us to remember that they also form a special set of numbers. This set of numbers is known as the **counting numbers** or **natural numbers** and consists of the elements 1, 2, 3, 4, . . . (the three dots mean "and so on in the same pattern").

2. Your earliest schoolwork in arithmetic probably involved the operation of addition with counting numbers. Later, you studied the operation of multiplication with counting numbers. Can we speak of a system whose elements are the counting numbers?

Is it possible to list *all* the names of the elements of this system? If so, we would call it a **finite** system. Is the system whose elements are counting numbers a finite system?

Do you think we may assume that addition in this system is both commutative and associative? May we assume that multiplication is commutative and associative?

3. Is 63 a counting number? Is 27? Is the sum of 63 and 27 a counting number? What is the sum of 77 and 86? of 8 and 117? In each case, is it true that the sum of two counting numbers is also a counting number? Do you agree that the set of counting numbers is closed under addition?

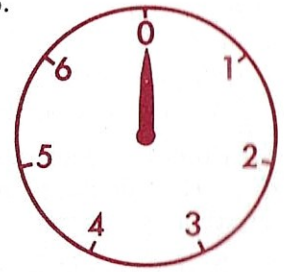
4. Is 7 a counting number? Is 2? What is 7 divided by 2? Is the quotient a counting number? Then the set of counting numbers is *not* closed under division.

5. If everyone suddenly left your classroom, how many people would be in the room? Suppose, in answering this question, you were allowed to use only a counting number. Could you answer the question? To answer, what non-counting number is necessary?

6. The number 0 and the counting numbers together form another special set of numbers. Do you think we may assume that addition is commutative and associative for this set of numbers? May we make the same assumption with regard to multiplication?

A Modular System

Thus far we have examined one finite system—the lever system. As you may already know, we can construct a finite system using an arbitrary number of elements drawn from zero and the counting numbers. For example, we can construct a system that utilizes just seven elements—0, 1, 2, 3, 4, 5, and 6.



Imagine a clockface that is used to indicate what day of the week it is. Suppose the hand is set at 0 on a given day—let us say Sunday. With each day that passes, the hand moves clockwise one interval. If six days pass and then another day passes, will it be Sunday again? That is, will the hand be at the starting point, 0, again?

Suppose six days pass, and then another two days pass. Will the net result (so far as the day of the week is concerned) be the same as though only 1 day had passed? That is, will it now be Monday? Will the hand be at the 1?

1. Let us now actually construct our system by defining an operation on the clockface numbers 0, 1, 2, 3, 4, 5, and 6. We will call the operation "addition" and denote it by the symbol $\oplus$ (read as "plus"). The operation will mean a counting of intervals clockwise along the dial. The beginning point is always zero.

   a. Count 6 intervals. Then count 1 more interval. Do you arrive back at 0? We write: $6 \oplus 1 = 0$.

   b. Count 6 intervals. Then count 2 more intervals. Do you arrive at 1? We write: $6 \oplus 2 = 1$.

2. What does $6 \oplus 3$ equal? $6 \oplus 4$? $6 \oplus 5$? $6 \oplus 6$?

3. The expression $6 \oplus 0$ may be interpreted as: Count 6 intervals, then count 0 intervals more. Does $6 \oplus 0$ equal 6?

$\oplus$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6	6	0	1	2	3	4	5

4. At the left is the form for a table in which the possible combinations for "addition" in this system can be indicated. The numerals at the left represent the first number named in an example; the numerals at the top represent the second number named in an example. The entries in the table, you can easily see, represent the results of $6 \oplus 0$, $6 \oplus 1$, $6 \oplus 2$, $6 \oplus 3$, $6 \oplus 4$, $6 \oplus 5$, and $6 \oplus 6$.

5. Make a similar table and enter the results of these examples.

a. $0 \oplus 6$ **b.** $1 \oplus 6$ **c.** $2 \oplus 6$
d. $3 \oplus 6$ **e.** $4 \oplus 6$ **f.** $5 \oplus 6$

Compare the results for $6 \oplus 0$ and $0 \oplus 6$; $6 \oplus 1$ and $1 \oplus 6$; and so on. Does it seem that the operation $\oplus$ is commutative?

6. Complete your table. Does the table show all the possible results of pairing elements under the operation $\oplus$? Can you now definitely say that the operation $\oplus$ is commutative?

7. Is the system closed under the operation $\oplus$? Explain.

▶ We call the system we have constructed a **modular** system. Since it has just seven elements, it is called a **modulo 7** system.

8. Using the table, determine what $(3 \oplus 2) \oplus 5$ equals. Determine what $3 \oplus (2 \oplus 5)$ equals. Is this sentence true: $(3 \oplus 2) \oplus 5 = 3 \oplus (2 \oplus 5)$?

You recognize that we are considering now whether the operation $\oplus$ is associative.

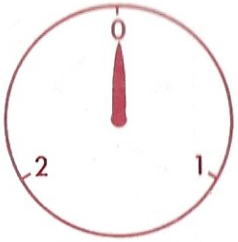
Are the sentences below true, or are they false?

a. $(4 \oplus 1) \oplus 3 = 4 \oplus (1 \oplus 3)$ **b.** $(2 \oplus 4) \oplus 1 = 2 \oplus (4 \oplus 1)$
c. $(3 \oplus 5) \oplus 2 = 3 \oplus (5 \oplus 2)$ **d.** $(4 \oplus 4) \oplus 6 = 4 \oplus (4 \oplus 6)$

Does it seem reasonable to assume that the operation $\oplus$, modulo 7, is associative?

The Operation $\oplus$ in Modulo 3

In the same way as the modulo 7 system was constructed using the elements 0, 1, 2, 3, 4, 5, and 6, a modulo 3 system can be constructed using the elements 0, 1, 2. You can easily investigate "addition" in modulo 3 on your own. The clockface at the right can be used the way the clockface for modulo 7 was used. The operation of "addition" is again defined as a counting of intervals clockwise beginning at 0.



Just to make sure that you understand the counting operation, what does $1 \oplus 2$ equal? $2 \oplus 2$? (If you obtained the answers 0 and 1, respectively, to the questions above, then you probably understand the operation.)

At the right is the form for a table for the operation $\oplus$ in modulo 3. Copy and complete the table so that it shows all possible results for pairings of elements in modulo 3.

$\oplus$	0	1	2
0			
1			
2			

EXERCISES

Complete Exercises **1–5** by referring to the table you completed for $\oplus$ in modulo 3.

1. Does the table define the operation $\oplus$ in modulo 3? In how many possible ways can you pair elements, using this operation?

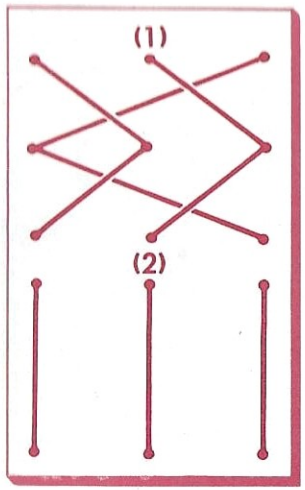
2. Is the operation commutative?

3. Does the system have the property of closure?

4. Using the table you have prepared, determine what $(1 \oplus 2) \oplus 2$ equals. Determine what $1 \oplus (2 \oplus 2)$ equals. Are each of the following sentences true?

a. $(1 \oplus 2) \oplus 2 = 1 \oplus (2 \oplus 2)$ **b.** $(2 \oplus 1) \oplus 0 = 2 \oplus (1 \oplus 0)$
c. $(1 \oplus 1) \oplus 2 = 1 \oplus (1 \oplus 2)$ **d.** $0 \oplus (2 \oplus 2) = (0 \oplus 2) \oplus 2$

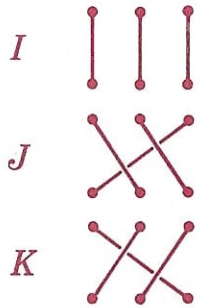
5. Is the operation $\oplus$, modulo 3, associative? Make up other examples if you are not sure.



A System without Numbers

1. Illustration (1) shows a pattern that results from stretching three elastic strips between pegs on a pegboard. (None of the strips are intertwined.) Imagine that the three pegs in the center row are removed and the strips contract. Can you see that the resulting pattern will be the pattern shown in illustration (2)?

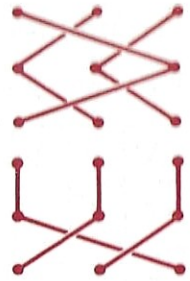
If you have difficulty seeing that this is so, copy the diagram in (1). Then draw a straight line segment from the peg to which each strip is fastened in the top row to the peg to which it is fastened in the bottom row.



2. Many different patterns may be formed on the pegboard with three elastic strips, but we will consider only the three patterns shown at the left. For convenience, we have named these patterns I, J, and K, respectively. Each of the patterns may be formed either between the middle row of pegs and one of the other rows, or it may be formed between the top row and the bottom row. Regardless of which two rows are used, the pattern is considered to be the same pattern.

In illustration (1), one of the patterns appears between the top and middle rows of pegs and a second between the middle and bottom rows. What is the name of the upper pattern? the lower? What pattern resulted from removing the middle row of pegs?

3. Let us consider the patterns I, J, and K as elements of a set. To construct a system, there must be an operation. In **1** above, we operated on the elements J and K by removing the middle row of pegs. So let us agree that the operation in this system is "remove the row of pegs between the two elements" and use the symbol "$\uparrow$" for this operation. (Read the symbol $\uparrow$ as "slip.") Then the situation in **1** will be represented as $J \uparrow K$. What does $J \uparrow K$ equal?

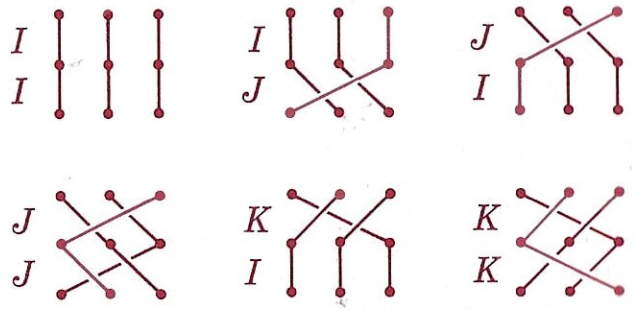


4. Suppose the positions of J and K were reversed on the pegboard as shown at the right. What pattern is obtained as the result of $K \uparrow J$? Compare the result with the result in **3**. Is the following a true sentence: $J \uparrow K = K \uparrow J$?

5. What is the upper pattern in this arrangement of the strips? What is the lower pattern? What pattern results when the middle row of pegs is removed? Is the following sentence true: $I \uparrow K = K$?

$\uparrow$	I	J	K
I			K
J			I
K		I	

6. At the right is a table for defining the $\uparrow$ operation in the I, J, K elastic-strip system. The letters at the left represent the element named first in an example. As you can see, the results of the operations described in the previous exercises have been entered in the table. Copy and complete the table. Refer to the diagrams below if you need help.



7. Is the I, J, K elastic-strip system closed under the operation $\uparrow$?

8. Is the operation $\uparrow$ always commutative in the I, J, K system? Use your completed table to see whether there are any exceptions.

9. On a pegboard with four rows of pegs, the strips might be arranged in three successive patterns. Use your table to find what $(I \uparrow J) \uparrow K$ equals; $I \uparrow (J \uparrow K)$.

10. Is each of these sentences true? Does it seem that the operation $\uparrow$ is associative?

a. $(J \uparrow I) \uparrow K = J \uparrow (I \uparrow K)$
b. $(K \uparrow J) \uparrow K = K \uparrow (J \uparrow K)$
c. $(I \uparrow K) \uparrow I = I \uparrow (K \uparrow I)$

A Generalized Mathematical System with Three Elements

$\oplus$	X	Y	Z
X	X	Y	Z
Y	Y	Z	X
Z	Z	X	Y

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\uparrow$	I	J	K
I	I	J	K
J	J	K	I
K	K	I	J

1. Each of the tables above defines an operation for a system you have studied. Identify each of the systems.

2. You can see by comparing the tables that the systems have certain similarities. How many elements are there in each system? Is each system closed under the operation? Is the operation in each system commutative?

3. In order to make a closer comparison of the three systems, let us make use of the symbols a, b, and c. Each symbol represents one of the elements of the three systems, as indicated at the right.

a represents X, 0, or I
b represents Y, 1, or J
c represents Z, 2, or K

Let us also make use of an operation symbol, $*$, which is read as "operate on." The $*$ symbol may represent the $\oplus$ operation of the lever system or the modulo 3 system, or the $\uparrow$ operation of the elastic-strip system.

Rewrite the sentence at the right, replacing the symbols with the appropriate symbols of the lever system. Is the sentence true for these replacements?

$a * b = b$

Rewrite the sentence again, using replacements from the modulo 3 system. Is the resulting sentence true? Is the sentence also true for replacements from the elastic-strip system?

Early in the chapter you saw how it was possible to use letters to write generalizations about numbers. You can now see that it is also possible to make generalizations that apply both to systems in which the elements are numbers and to systems in which the elements are not numbers.

4. Is each of the following sentences true for replacements from the lever system? the modulo 3 system? the elastic-strip system?

a. $a * a = a$ **b.** $a * c = c$ **c.** $b * b = c$

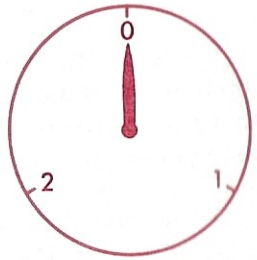
5. The table at the right defines the operation $*$ for the elements a, b, and c. Since the elements and the operation have a special meaning only when they are replaced, we say that the a, b, c system is a **generalized mathematical system.**

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

6. You know that the elements and the operation of the lever system have certain specified meanings. Is this also true of the modulo 3 system and the elastic-strip system? Mathematicians say that each of these systems is a *model* of the generalized a, b, c system.

The elements and operation of the generalized system may represent the elements and operations of the three models you have studied, but they can also represent the elements and operation of many other models.

Another Operation in Modulo 3



1. Let us now consider a second operation in modulo 3. We will use the symbol ⊗ and we will read this symbol as "times," since the operation will be similar to multiplication in arithmetic. That is, 1 ⊗ 2 will mean that, beginning at zero, we are to count clockwise (or move the hand of the clock) one interval of 2 units. What does 1 ⊗ 2 equal?

2. To find the "product" 2 ⊗ 2, we begin at 0 and count clockwise two intervals of 2 units. What does 2 ⊗ 2 equal?

3. We interpret 0 ⊗ 2 as the counting of zero intervals of 2 units. What does 0 ⊗ 2 equal?

4. At the right is the form for a table for defining the operation ⊗ in modulo 3. The numbers named at the left represent the "multipliers"; that is, the first-named element in each pair. Thus, the entry in the table represents the result for 1 ⊗ 2. Copy and complete the table.

⊗	0	1	2
0			
1			2
2			

5. Does your completed table define the operation ⊗ in modulo 3? Is the operation commutative? Is the modulo 3 system closed under the operation ⊗?

6. Use the table to determine whether or not each of the following sentences is true: $(1 \otimes 2) \otimes 2 = 1 \otimes (2 \otimes 2)$; $(2 \otimes 0) \otimes 1 = 2 \otimes (0 \otimes 1)$; $(2 \otimes 1) \otimes 1 = 2 \otimes (1 \otimes 1)$. Does it seem that the operation ⊗, modulo 3, is associative?

A Property That Relates Two Operations

⊕	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

⊗	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Above are the tables that define the operations ⊕ and ⊗ in the modulo 3 system. The modulo 3 system has an important property that relates the operations ⊕ and ⊗. Try to discover this property for yourself as you complete the work below.

Using the tables (and remembering that operations within parentheses are to be performed first), complete the indicated operations in **1a** and **b**; **2a** and **b**; **3a** and **b**.

	a	b
1.	$2 \otimes (1 \oplus 2)$	$(2 \otimes 1) \oplus (2 \otimes 2)$
2.	$2 \otimes (1 \oplus 0)$	$(2 \otimes 1) \oplus (2 \otimes 0)$
3.	$1 \otimes (2 \oplus 2)$	$(1 \otimes 2) \oplus (1 \otimes 2)$

In each pair, did you obtain the same result in part **b** as in part **a**? Then are the sentences at the right true?

$$2 \otimes (1 \oplus 2) = (2 \otimes 1) \oplus (2 \otimes 2)$$
$$2 \otimes (1 \oplus 0) = (2 \otimes 1) \oplus (2 \otimes 0)$$
$$1 \otimes (2 \oplus 2) = (1 \otimes 2) \oplus (1 \otimes 2)$$

Try to describe in your own words what these sentences indicate.

Without performing the operations, do you think that $2 \otimes (0 \oplus 2) = (2 \otimes 0) \oplus (2 \otimes 2)$? We say that in modulo 3 the operation ⊗ is **distributive** over the operation ⊕.

4. Perhaps you recognized that it was the distributive principle you were considering. However, since the modulo 3 system is somewhat unfamiliar to you, you may find it helpful to see how the property of distributivity applies to operations in the more familiar system of the counting numbers.

For each sentence below, complete the operations on both sides of the equal sign. Show all the steps in your work.

a. $7 \times (3 + 5) = (7 \times 3) + (7 \times 5)$
b. $2 \times (8 + 17) = (2 \times 8) + (2 \times 17)$
c. $6 \times (22 + 36) = (6 \times 22) + (6 \times 36)$
d. $9 \times (11 + 12) = (9 \times 11) + (9 \times 12)$

5. Your work in **4a–d** indicates that in the set of counting numbers multiplication is distributive over addition. This means that, to find the product of a number and the sum of two numbers, you may first add and then multiply or you may multiply each addend by the multiplier and then find the sum of the products. Did the preceding sentence seem difficult to understand? Some mathematical ideas—even some fairly simple ones—cannot easily be expressed in words. One of the great advantages of algebra is that through the use of symbols it permits us to express mathematical ideas in a simple form. For example, you should be able to see that for all counting numbers that are used as replacements for a, b, and c the following generalization is true.

$$a \times (b + c) = (a \times b) + (a \times c)$$

6. Is the generalization shown above true for all replacements from the numbers of arithmetic? That is, is multiplication distributive over addition for the numbers of arithmetic?

EXERCISES

In each of the following exercises, find the number of arithmetic that you can use as a replacement to make the sentence true.

1. $5 \times (2 + 11) = (5 \times 2) + (5 \times r)$
2. $6 \times (1\frac{1}{4} + r) = (6 \times 1\frac{1}{4}) + (6 \times \frac{1}{4})$
3. $2 \times (3\frac{1}{2} + 6\frac{1}{2}) = 2 \times h$
4. $(4 \times 6) + (3 \times 6) = y \times 6$
5. $1\frac{1}{2} \times (2 + k) = 3 + 3$
6. $h \times (\frac{1}{4} + \frac{3}{4}) = 1 + 3$
7. $(m \times 5) + (2 \times 5) = 6 \times 5$

Properties of a System

1. A mathematical system is composed of elements and at least one operation. The elements may be numbers; *must* they be?

2. Any system under one operation may have certain properties. You have studied three of these properties. They are listed in the table at the right. Write a mathematical sentence that illustrates each property. You may choose any set of numbers and any operation that has these properties.

Properties that a system may have under an operation
The operation is commutative.
The operation is associative.
The system is closed under the operation.

3. You have also learned of a property that relates two operations involving the elements in a system. What is the name of this property? Write a sentence that illustrates the property. You may choose your illustration from any system you know.

SETS AND SENTENCES

Representing Sets

1. The word *set* has been used earlier in this chapter, and you have probably had some experience in working with sets in your previous courses in mathematics. As a result, you should have little difficulty in naming all the members or elements of each of the following.

a. the set of counting numbers smaller than 10
b. the set of people who live in your home

2. Could you name the elements of the set of all pretty colors? Would your list necessarily be the same as the list of each of your classmates? As the word *set* is used in mathematics, there is no such set as "the set of all pretty colors," because the word "pretty" does not exactly describe the property a color must have in order to be a member of the set. Which of the following could be regarded as a set in mathematics?

a. all lucky numbers
b. 0, 1, 2, 3, 4, 5
c. all large numbers
d. some numbers smaller than 10
e. all even numbers greater than 6
f. the numbers used in the modulo 3 system
g. 10, 11, 12
h. the letters a, e, i, o, u

In mathematics, the word *set* is used to refer to a well-defined collection. Everyone must be able to tell from the definition what *is* a member of the set and what *is not* a member of the set.

3. For convenience, braces are used in mathematics to designate a set. Thus, the set {all numbers greater than 10} denotes the set of all numbers greater than 10. A set may also be designated by a capital letter. For example, if we wrote: $Q =$ {all numbers greater than 10}, we would be saying that any time we mention set Q we mean the set of all numbers greater than 10.

For each set in **2**, write a sentence in which you designate the set by a capital letter. Use the braces in each sentence.

4. Two of your answers in **3** may have been as follows:

$B = \{0, 1, 2, 3, 4, 5\}$

$E =$ {all even numbers greater than 6}

Notice that sets B and E are defined in different ways.

Set B is defined by listing *all* of its elements. Is 3 a member of set B? Is 19?

Set E is defined by stating the condition necessary for membership in the set. What is this condition? Is 8 a member of set E? Is 15? Is 222? Is 3.4?

We say that set B is defined by **tabulation** and that set E is defined by **description.**

5. Tell which of the other sets you defined (in **3**) were defined by tabulation; that is, by listing *all* of the members in the set. Tell which of the other sets were defined by description; that is, by stating the condition necessary for membership.

6. You know that some of the numbers in the set of counting numbers are odd and some are even. Do the odd counting numbers form a set? Do the even counting numbers form a set?

We say that the set of odd counting numbers is a **subset** of the counting numbers because every element in the set of odd counting numbers also belongs to the set of counting numbers.

Of what set of numbers is the counting numbers a subset?

7. The set of all odd counting numbers can be defined by description as follows: $X = \{\text{all odd counting numbers}\}$. Does this set have an unlimited number of members? Would it be possible to list *all* the members of the set? Then, strictly speaking, we cannot define the set by tabulation. However, we can *adapt* the method of tabulation and define the set as follows:

$$X = \{1,\ 3,\ 5,\ 7,\ \ldots\}$$

Use the same method to define the set consisting of the even counting numbers.

EXERCISES

Define sets *A*–*D* by description; define sets *E*–*H* using the method of tabulation.

$A = \{20, 21, 22, \ldots\}$
$B = \{5, 7, 9, \ldots\}$
$C = \{\text{George Washington, John Adams}\}$
$D = \{\text{a, b, c, d, e}\}$
$E = \{\text{all even counting numbers less than 10}\}$
$F = \{\text{all counting numbers less than 9}\}$
$G = \{\text{the numbers used in the modulo 7 system}\}$
$H = \{\text{the days of the week}\}$

Inequality Symbols

1. Do you know the meaning of the symbols "$<$" and "$>$" in the sentences at the right? These two inequality symbols are used in sentences much as "$=$" is used, but of course they have different meanings. The first sentence is read "2 is less than 3." Is this sentence true? The second sentence is read "6 is greater than 1." Is this sentence true?

$2 < 3$
$6 > 1$

Consider the sentence $9 < 7$. How do you read the sentence? Is the sentence true?

2. Express the following as mathematical sentences using the inequality symbols "$<$" and "$>$." Then tell which of the sentences are true and which are false.

a. Seven is greater than two.
b. Thirty-six is less than twenty-five.
c. Five is greater than eleven.
d. Fifteen is less than sixteen.

3. Another inequality symbol appears in the sentence at the right. This sentence is read "a is less than or equal to 9."

$a \leq 9$

Suppose a represents any number in the set of counting numbers. Name two replacements for a that will make the sentence true. Name one replacement for a that will make the sentence false.

4. The sentence $n \geq m$ is read "n is greater than or equal to m." Suppose n and m represent counting numbers and n is replaced by 4. Name one replacement for m that will make the sentence true. Name two replacements for m that will make the sentence false.

$n \geq m$

5. The symbol "$\neq$" is read as "is not equal to." Read the model sentence. Suppose r represents any number in the set of counting numbers. Name one replacement that will make the sentence true. Name one replacement that will make the sentence false. Is there any replacement other than the number 6 that will make the sentence false?

$r \neq 6$

6. Express each of the following as a mathematical sentence using one of the inequality symbols $<$, $>$, $\leq$, $\geq$, $\neq$.

a. n is greater than or equal to 1.
b. y is less than 5.
c. a is less than or equal to 7.
d. g is greater than 21.4.
e. Fifteen is greater than q.
f. x is not equal to 3.2.

EXERCISES

1. Tell which of the following sentences are true and which are false.

a. $3 < 11$
b. $16 < 2$
c. $3 + 6 = 8$
d. $6 > 7$
e. $8.2 < 8.3$
f. $12.010 > 12.009$

2. Write each of the following as a mathematical sentence using one of these symbols: $<$, $>$, $\leq$, $\geq$, $\neq$.

a. q is less than 15.
b. r is greater than or equal to 33.
c. s is greater than $12\frac{1}{2}$.
d. t plus 3 is not equal to 5.
e. Four is less than m.
f. n is less than or equal to 7.2.

3. Each of the symbols, q, r, s, . . ., used in Exercise **2** represents any number in the set of numbers of arithmetic. For each sentence, name one replacement that will make the sentence true and one replacement that will make the sentence false.

Variables and Replacement Sets

Consider the sentence at the right. Is it a true sentence as it stands? Is it false? Do you agree that it is neither true nor false until we replace a by the name of some number?

$$a + 7 < 10$$

As you can see, the symbol a may be thought of as a **placeholder** for the name of a number. So far in this book, we have used no special name for a symbol such as a that is used as a placeholder. Such a symbol is called a **variable.** A key idea in algebra is that a variable cannot be replaced until there is an understanding about the set of numbers from which replacements may be chosen.

The set of numbers from which replacements for a variable may be chosen is called the **replacement set** for the variable.

Suppose you are told that a represents a counting number. Then the replacement set for a is the set $\{1, 2, 3, \ldots\}$. If, on the other hand, you are told that a represents any odd counting number, how would you describe the replacement set for a?

If you are given no information about what a variable such as x, y, or a represents, it is understood that the replacement set is the set of all the numbers you know.

EXERCISES

1. If n represents the number of students in a class, could 30.4 be one of the possible replacements for n? Explain your answer.

2. If y represents an element in the modulo 7 system, how would you represent the replacement set for y?

3. If x represents an element in the modulo 3 system, what is the replacement set for x?

4. Suppose that during the course of a game a basketball player made more than 5 field goals and scored a total of n points but did not make any foul shots. How would you represent the replacement set for n? (There is not just one "correct" answer to this question, but one answer is a bit "neater" than the others. Test your imagination.)

5. Suppose y represents any member of the set of leap years between the years 1801 and 1899. Does the replacement set for y include the year 1820? 1835? 1842? 1860? 1875? 1888? How many year numbers are in the replacement set?

Open Sentences and Statements

1. Suppose the set $\{1, 2, 3, 4, 5, 6\}$ is the replacement set for a in the sentence $a + 7 < 10$. Which of the replacements will make the sentence true? Which will make the sentence false?

2. For the sentence at the right, let the replacement set for x be the set $\{1, 2, 3, 4, 5, 6\}$. Which of the possible replacements will make the sentence true? Which will make it false?

$$x + 3 = 5$$

3. As you have seen, a sentence such as $x + 3 = 5$ or $a + 7 < 10$ is neither true nor false until the variable is replaced by the name of a number. In mathematics, a sentence that is neither true nor false is called an **open sentence.** When a sentence is either definitely true or definitely false, it is called a **statement.** Thus, when x is replaced by 2, we obtain the true statement $2 + 3 = 5$. When x is replaced by 3, we obtain the false statement $3 + 3 = 5$.

Tell which of the following are open sentences and which are statements. If a sentence is a statement, tell whether it is a true statement or a false statement.

a. $15 + 17 = 32$	**b.** $64 > 88$	**c.** $m - 5 = 15$
d. $27 + 36 = 61$	**e.** $s + 8 > 16$	**f.** $43 < 47$
g. $19 + n = 28$	**h.** $84 - 22 = 56$	**i.** $30 + t < 40$

Solution Sets

When we find the replacement or replacements for a variable that make an open sentence a true statement, we say that we have **solved** the sentence or found the **solution** of the sentence.

1. Suppose the set $\{1, 3, 5, 7, 9\}$ is the replacement set for y in the sentence at the right. Does a true statement result when y is replaced by 1? by 3? by 5? by 7? by 9?

$$7 > y + 3$$

Then the numbers 1 and 3 are said to be solutions of the sentence, or *the* solution of the sentence. Since the solution itself forms a set, we call the set $\{1, 3\}$ the **solution set**, or **truth set**, of the sentence.

2. For the sentence at the right, let the replacement set be the set $\{1, 3, 5, 7, 9\}$. Is 1 a solution of the sentence? 3? 5? 7? 9? Then the solution set is $\{7\}$, a set with just one member.

$$6 = r - 1$$

3. Look back at your work on page 23. Tabulate the solution set of $a + 7 < 10$. Tabulate the solution set of $x + 3 = 5$.

4. The replacement set for the variable in each of the following sentences is the set $\{0, 1, 2, \ldots, 8\}$. The three dots here indicate that some members have not been listed; the "8" shows that 8 is the final member in the pattern indicated. Find the solution set of each sentence.

a. $s + 3 = 8$ **b.** $t + 2 < 4$ **c.** $g + 7 = 14$ **d.** $k + 7 < 13$
e. $p + 8 = 8$ **f.** $m + 5 > 9$ **g.** $r + 5 = 13$ **h.** $11 + n > 16$

5. Suppose the set $\{0, 1, 2, 3, \ldots\}$ is the replacement set for the variable y in the sentence at the right. Is there a replacement that will make the sentence a true statement?

$$y + 1 = 4.5$$

When there are no elements in a replacement set that will make a sentence a true statement, we say that the solution set is the **empty set.** The empty set, which is also called the **null set**, has no elements and is usually represented by the symbol "$\emptyset$." No braces are used with the symbol.

6. Let the set $\{2, 4, 6, 8\}$ be the replacement set for the variable in each of the following sentences. Find the solution set of each sentence. If the solution set is the empty set, use the symbol $\emptyset$.

a. $x + 2 = 8$ **b.** $y + 2 = 9$ **c.** $7 - 2 = a$ **d.** $8 - z < 4$
e. $m + 8 > 9$ **f.** $s + 16 = 24$ **g.** $n + 4 < 5$ **h.** $t + 18 = 20$

EXERCISES

The replacement set for the variable in each of the following sentences is $\{1, 3, 5, 7, 9\}$. Find the solution set of each sentence.

1. $x + 3 = 4$ **2.** $m + 3 > 6$ **3.** $y - 1 = 8$
4. $n - 1 > 4$ **5.** $r + 4 = 10$ **6.** $c + 10 < 20$
7. $s + 9 = 12$ **8.** $d + 12 < 14$ **9.** $t + 8 = 15$

The replacement set for the variable in each of the following sentences is $\{2, 4, 6, 8, 10\}$. Find the solution set of each sentence.

10. $p + 15 = 19$ **11.** $w + 4 < 11$ **12.** $q + 23 = 31$
13. $x + 6 < 15$ **14.** $r + 17 = 27$ **15.** $y + 14 > 21$
16. $s + 9 = 14$ **17.** $z - 2 > 3$ **18.** $t - 2 = 4$

Working with Open Sentences

In algebra, a few of the familiar symbols of arithmetic are not usually used. For example, "$\times$" is rarely used to represent multiplication. Can you suggest one reason why this should be so?

1. In place of $\times$, a center dot "$\cdot$" is often used, as $7 \cdot 8 = 56$. Rewrite the following open sentences, using "$\cdot$" in place of $\times$. Then rewrite each open sentence as a true statement, using the numbers of arithmetic as the replacement set for each variable.

a. $3.2 \times 8.6 = x$ **b.** $5\frac{1}{4} \times 2\frac{1}{3} = y$ **c.** $z = 100 \times 1000$

2. When indicating that the product of a number and the replacement for a variable is to be found, no symbol is used. For example, instead of $5 \times y$ or $5 \cdot y$, the usual algebraic notation is $5y$. This is read "five y" and means that any replacement for y is to be multiplied by 5. Tabulate the solution set of each of the following open sentences. The replacement set is the set of counting numbers.

a. $3y < 9$ **b.** $2r + 3 = 7$ **c.** $5m = 10$

3. When a symbol appears directly before a group of symbols that are enclosed in parentheses, multiplication is also indicated. For example, $7(3 + 10)$ indicates that we are to find the product of 7 and the sum of 3 and 10; $x(4 - 2)$ indicates that we are to find the product of any replacement for x and the difference between 4 and 2.

Express each of the following in symbols.

a. two less than the product of 4 and y
b. the product of 4 and the number that is 2 less than y

4. In each of the following open sentences, find a replacement for the variable that will make the sentence a true statement. The replacement set in each case is the set of numbers of arithmetic.

a. $x(4 - 2) = 10$ **b.** $2(x + 1) = 8$ **c.** $x(4 + 3) = 49$

5. Sometimes a variable appears more than once in a sentence. In such a case, when we replace the variable by a numeral, we must use the same replacement at each place the variable appears. If the replacement set for r is the set of counting numbers, find the solution set of each of the sentences below.

a. $r + r = 8$ **b.** $r + r + 6 = 12$ **c.** $10 = r + r + r - 2$

6. When an open sentence contains two variables, each may have its own replacement set. More frequently, however, the replacement set for both variables is the same set of numbers. In this case, it is not necessary to choose different numbers as replacements for the variables. For example, suppose the set of counting numbers is the replacement set for the variables a and b in the open sentence below.

$$a + b = 10$$

If we replace a by 6, what replacement for b will make the sentence a true statement? If we replace a by 5, what replacement for b will make the sentence a true statement? Find as many pairs of replacements as you can that will make the sentence a true statement.

EXERCISES

The replacement set for the variable in each of the following sentences is the set of counting numbers. Find the solution set of each sentence.

1. $6h = 30$	**2.** $9i = 27$	**3.** $4x < 16$	**4.** $3k = 18$
5. $8m = 64$	**6.** $5y < 25$	**7.** $15n = 45$	**8.** $7w = 84$
9. $3x + 2 = 11$	**10.** $5y + 4 = 24$	**11.** $4z + 4 < 18$	
12. $2e - 3 = 23$	**13.** $8f - 1 = 71$	**14.** $6g + 2 < 28$	
15. $r(9 - 4) = 40$	**16.** $s(6 + 5) = 77$	**17.** $t(3 + 7) = 120$	
18. $d + d + 14 = 20$	**19.** $n + n + n = 51$	**20.** $x + x - 2 = 24$	

Set Notation

1. Study the open sentence at the right. Suppose the replacement set for x is the set of numbers of arithmetic. Will we obtain a true statement if we replace x by $\frac{1}{2}$? by 1? by 2? by 2.1? by 2.01? Do you see that every number that is greater than 2 will be a member of the solution set?

$$3x > 6$$

Suppose we name the solution set A. We can use set notation and define set A as shown.

$$A = \{x \mid x > 2\}$$

The way in which this sentence is read is shown in the diagram below. Notice that the vertical bar is read as "such that."

$$\underbrace{\text{Set } A \text{ is the set of all } x \text{ such that } x \text{ is greater than 2.}}_{A \quad = \quad \{ \quad x \quad | \quad x > 2\}}$$

This is simply another way of saying that

$$A = \{\text{all numbers greater than 2}\}$$

2. Suppose the set of numbers of arithmetic is the replacement set for a in the sentence at the right. Will we obtain a true statement if we replace a by 3? by 4? by 4.01? by $4\frac{1}{15}$? Will every number that is greater than 4 be a member of the solution set? Do you see that set B, defined below, is the solution set of the sentence $20 < 5a$?

$$20 < 5a$$

$$B = \{a \mid a > 4\}$$

3. If the replacement set for m is the set of numbers of arithmetic, which of the following is the solution set of the sentence $2m + 2 < 6$?

$$C = \{m \mid m < 4\} \qquad D = \{m \mid m < 2\}$$

EXERCISES

The replacement set for the variable in each of the following sentences is the set of numbers of arithmetic. Use set notation to show the solution set of each sentence.

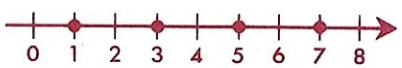
1. $7m < 21$	**2.** $9n > 45$	**3.** $8x < 60$
4. $12y > 48$	**5.** $3z < 39$	**6.** $4w > 34$
7. $p + 2 < 8$	**8.** $q + 15 > 18$	**9.** $r + 2\frac{3}{4} < 7\frac{3}{4}$
10. $s + 13 > 15\frac{1}{2}$	**11.** $t + 3\frac{1}{2} < 6\frac{3}{4}$	**12.** $u + 5.8 > 12.3$
13. $2d + 3 < 9$	**14.** $5e + 25 > 50$	**15.** $6f + 22 < 34$

The Graph of a Set

In algebra—as in all mathematics—graphs are of great value. Graphs often provide a means of representing algebraic ideas in a simple visual form.

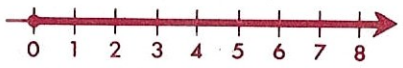
1. From your previous courses in mathematics, you are familiar with the way in which numbers of arithmetic are associated with points on a number line. In the figure at the right, the points corresponding to 0, 1, 2, 3, and 4 have been indicated. Where would you locate the point that corresponds to $\frac{1}{2}$? to $\frac{3}{2}$? to $3\frac{1}{4}$?

2. A number line can be very useful in picturing a set of numbers. For example, suppose we want a picture of the set of odd counting numbers; that is, we want a graph of the set $\{1, 3, 5, \ldots\}$. Corresponding to each number is a point; thus, the graph will be a set of points. The graph of the set $\{1, 3, 5, \ldots\}$ is shown below.



Notice that dots are marked at the points corresponding to 1, 3, and 5 to indicate that these are members of the set. Why is there a dot marked at the point corresponding to 7? Why is there no dot marked at the point corresponding to 8? The arrow is used to indicate that the graph extends indefinitely to the right.

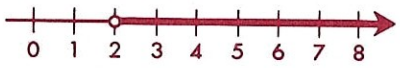
3. Draw a number line, and on it draw a graph of the set of even counting numbers, $\{2, 4, 6, \ldots\}$.



4. Suppose we wish a graph of the set of all the numbers of arithmetic. You know that this set includes zero and all numbers larger than zero that are used in arithmetic. You will later learn of a set of numbers larger than zero that is not a subset of the numbers of arithmetic, but for the present we will assume that the numbers of arithmetic include zero and all numbers greater than zero.

The picture above is the graph of the set $\{x \mid x \geq 0\}$. The dot at zero indicates that zero is included in the graph. The thickening of the number line to the right of zero is used to show that all the numbers that correspond to the points on this portion of the line also belong to the set. The arrow indicates that the graph continues indefinitely to the right.

5. One important use of graphs in algebra is in picturing the solution sets of open sentences. In 1 on page 26, you found that the solution set of $3x > 6$ was $A = \{x \mid x > 2\}$. The graph of set A is shown here. We mark a hollow dot at the point corresponding to 2 to indicate that 2 is not a member of the set. What does the heavy line indicate? the arrow?

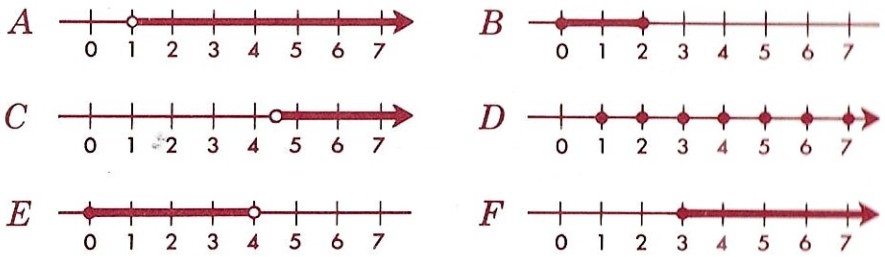


6. Below is a graph that consists of a single point. Is it the graph of the solution set of $3x = 9$? (The replacement set is the numbers of arithmetic.) Explain.

EXERCISES

1. Suppose that the sets listed in **a–f** below represent solution sets of open sentences. (Assume that the replacement set was the set of numbers of arithmetic.) Match each of the sets in **a–f** with the capital letter of the graph that corresponds to the set.

a. $\{x \mid x \geq 3\}$ **b.** $\{y \mid y > 1\}$ **c.** $\{z \mid z < 4\}$
d. $\{m \mid m > 4\frac{1}{2}\}$ **e.** $\{n \mid n \leq 2\}$ **f.** $\{1, 2, 3, \ldots\}$

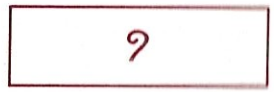


2. Draw a graph of the solution set of each of the following sentences. The replacement set is the set of numbers of arithmetic.

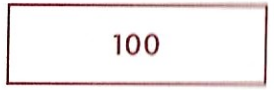
a. $p + 2 > 5$ **b.** $5q \geq 10$ **c.** $4x = 12$
d. $r + 7 < 12$ **e.** $3y = 15$ **f.** $4s \geq 16$
g. $z + 3 = 4$ **h.** $t + 8 \geq 14$ **i.** $u + 9 \leq 9$

NUMBERS AND NUMBER RELATIONSHIPS

Numerals: The Names for Numbers



The curly line in the box at the right probably has no meaning to you. As a mathematical symbol, it therefore has no value to you. To a scholar in ancient Egypt, the symbol "100" would be just as meaningless.



But both symbols are names for the same number and are called **numerals.** Any symbol or group of symbols that stands for a number is a numeral. Thus, a numeral is a name of a number.

Do these expressions name the same number? How do we ordinarily represent this number? The numeral "101" is the **common name** for the number 101. Give the common names for the following.

$50 + 51$
$106 - 5$
2×50.5

a. $13 + 122$ **b.** $\frac{52}{13}$ **c.** $17.5 - 3.5$ **d.** 12×12

Exponents

1. Frequently in mathematics you will wish to express a number in **exponential form**; that is, in the form of the numeral shown at the right. This numeral is read as "seven to the third power" or "seven cubed."

7^3 (3: exponent, 7: base)

What does the small "3" mean in this numeral? You probably recall that an exponent tells the number of times the base is to be used as a factor. In this case the "3" means that 7 is to be used as a factor three times: $7^3 = 7 \cdot 7 \cdot 7$. What is the product?

You can see that the common name for 7^3 is 343. We describe the relationship between 7 and 343 by saying that 343 is the **third power** of 7 or the **cube** of 7. What is the common name for 7^2? We say that this number is the **square** of 7. What is the common name for the fourth power of 7? What is the common name for $(1.1)^2$? $(\frac{1}{2})^2$?

2. Rewrite each of the following expressions to show how many times the base is to be used as a factor. Then write the common name for each number. Show your work in the form of the model at the right.

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2$
$= 16$

a. 123^2 **b.** 5^6 **c.** 4^4 **d.** 15^2 **e.** $(1.5)^2$ **f.** 3^7

3. Suppose we use the variable a to represent any member of the set $\{1, 2, 3, 4\}$. What would a^3 equal if a were replaced by 1? by 2? by 3? by 4?

4. Suppose the replacement set for the variable z is the set $\{5, 10, 15\}$. What would z^3 equal if z were replaced by 5? by 10? by 15?

5. You have seen that variables are used to represent numbers. Thus, we may interpret the expression a^3 as $a \cdot a \cdot a$. How would you interpret x^5? y^4?

6. You know that 3^3 means $3 \cdot 3 \cdot 3$, and 3^2 means $3 \cdot 3$. What meaning do you think it might be reasonable to give to 3^1?

Since 7^3 means $7 \cdot 7 \cdot 7$, and 7^2 means $7 \cdot 7$, do you think it might be reasonable to define 7^1 as 7? 8^1 as 8? a^1 as a?

EXERCISES

The variable in each of the expressions in the table below represents any member of the set of numbers of arithmetic. First show what the expression means, using the multiplication symbol. Then make the indicated replacement for the variable and find the product. Copy and complete the table.

Expression	Meaning	If replacement is:	Then:
y^2	$y \cdot y$	3	$y^2 = 9$
z^3	?	8	$z^3 =$?
x^8	?	2	$x^8 =$?
q^2	?	100	$q^2 =$?
a^2	?	6.5	$a^2 =$?
c^3	?	11	$c^3 =$?

Expressions

Thus far, the word "expression" has been used without explanation. In mathematics, *expression* does have a specific meaning, and it is worthwhile to establish exactly what this meaning is.

In mathematics, an expression is any one of the following:	Examples
A number	3, 1110, $.05$, $\frac{1}{8}$, 10^5
A variable	x, a, n
The sum of two expressions	$x + y$, $7 + x$
The difference of two expressions	$z - y$, $8 - x$
The product of two expressions	$3x$, xy, y^2, $5(x + y)$
The quotient of two expressions	$\frac{x}{y^2}$, $\frac{z+3}{2}$

1. In the sentence $7x + 3 = y$, is "$7x + 3$" an expression, according to the definition given above? Is "y" also an expression? In the sentence $4 > 3 + x$, is "4" an expression? Is "$3 + x$" also an expression? Do you think it is true that every mathematical sentence must consist of at least two expressions?

2. Write each of the following in mathematical symbols. Tell which are sentences and which are expressions, not sentences.

a. the sum of 3 and x
b. x divided by y equals z
c. the product of 7 and y^2
d. 9 is less than x
e. 4 subtracted from z equals 2
f. z divided by x

3. On pages 30–31, you made replacements for the variable in expressions such as a^3, y^2, and so on, and found the number named by the expression for each replacement. For example, when you used 4 as the replacement for a in the expression a^3, you found that a^3 was equal to 64. We call 64 the **value** of the expression a^3 **at** 4, and we say that we have **evaluated** the expression at 4.

What number is named by the expression $2(z + 3)$ if you replace z by 5? Then the value of $2(z + 3)$ at 5 is $\underline{?}$.

Evaluate each of the following at 6.

a. z^2 **b.** $\frac{4 + y}{2}$ **c.** $x + x^2$ **d.** $r^3 + 3r$

EXERCISES

1. Evaluate the following expressions at 4.

a. $x^2 + 7$ **b.** $19 - x$ **c.** $\frac{10x}{3}$ **d.** $3(x + 2)$

2. Evaluate the following expressions at 7.

a. $\frac{y + 17}{3}$ **b.** $2(y^2 + 3)$ **c.** $123 - y$ **d.** $y + 8\frac{1}{2}$

3. Evaluate the following expressions at 3.

a. $z^2 + 3z$ **b.** $\frac{z^3 + z}{5}$ **c.** $z^2 - 9$ **d.** $2(z^4 - 2z)$

Multiplication and Exponents

Exponents are used in science and mathematics. Let us study some patterns of operations with numbers in this form.

1. How many times is 5 used as a factor in the expression at the right? Then how is the product expressed in exponential form?

$5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

2. Is it true that $5^3 = 5 \cdot 5 \cdot 5$? $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$? Then can we express $5^3 \cdot 5^4$ as $(5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5)$? How many times is 5 used as a factor? Then can we express the product of 5^3 and 5^4 in exponential form as 5^7? Shown here are the steps that were followed.

$$5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5)$$
$$= 5^7$$

3. Using the method shown in **2** and writing your work in the same form as the model solution, express the product of each of the following in exponential form.

a. $5^2 \cdot 5^3$ **b.** $4^4 \cdot 4^5$ **c.** $6^6 \cdot 6^3$ **d.** $10^5 \cdot 10^2$

4. Examine your written solutions for **3**. In each case, is there a relationship between the exponents in the original expressions and the exponent in the product? State this relationship in your own words.

On the basis of your work for **3**, would you say that the sentence at the right is true for all replacements for x? Check your answer by evaluating $x^5 \cdot x^4$ and x^9 at 2; at 3.

$$x^5 \cdot x^4 = x^{(5+4)}$$
$$= x^9$$

5. Study the sentence at the right. Do you think that this sentence is true for all replacements for x and y? Check your answer by replacing x by 2 and y by 3 and carrying out the indicated operations. What do you conclude about finding the product of two numbers in exponential form when the bases are not equal?

$$x^3 \cdot y^2 = (xy)^5$$

6. What does 0^2 equal? 0^3? 0^5? 0^9? Then $0^n = 0$, when n is a counting number.

7. What is the common name for $5^2 \cdot 5$? the common name for 5^3? Is this sentence true: $5^2 \cdot 5 = 5^3$?

In terms of exponents, we might consider the expression $5^2 \cdot 5$ as $5^2 \cdot 5^1$. Then, following the pattern of multiplication involving exponents, we may write:

$$5^2 \cdot 5^1 = 5^{(2+1)}$$
$$= 5^3$$

Does the agreement we made concerning an exponent of "1" seem to fit the pattern in multiplication?

Consider a few more examples, such as $3^4 \cdot 3$; $6 \cdot 6^2$; $a^3 \cdot a$.

8. Do you think that if m and n are counting numbers, and a is a number of arithmetic, the following sentence is true?

$$a^m \cdot a^n = a^{m+n}$$

Test your conclusion by choosing replacements for a, m, and n.

Division and Exponents

1. You have seen that to find the product of two powers with the same bases you may write the base with the sum of the exponents. For instance, you have found that $3^3 \cdot 3^2 = 3^5$. Can you reach a conclusion regarding a simple procedure for finding the quotient of $3^5 \div 3^2$? Do you think that the sentence at the right is true?

$$\frac{3^5}{3^2} = 3^3$$

You may check your answer regarding the sentence by the steps shown in the model solution at the right.

$$\frac{3^5}{3^2} = \frac{\overset{1}{\cancel{3 \cdot 3} \cdot 3 \cdot 3 \cdot 3}}{\underset{1}{\cancel{3 \cdot 3}}}$$

$$= \frac{3 \cdot 3 \cdot 3}{1}$$

$$= 3 \cdot 3 \cdot 3$$

$$= 3^3$$

You know that 3^5 may be expressed as $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ and that 3^2 may be expressed as $3 \cdot 3$. You can see that $(3 \cdot 3)$ is a factor common to both expressions, and when the expressions are divided by this common factor the result is $\frac{3 \cdot 3 \cdot 3}{1}$, or 3^3.

2. Find, in exponential form, the quotient of each of the following. Check your work by the method shown above.

a. $\frac{5^7}{5^3}$ **b.** $\frac{8^4}{8^2}$ **c.** $\frac{9^5}{9^4}$ **d.** $\frac{3^5}{3^3}$

3. On the basis of your work for **2**, would you say that the sentence at the right is true for all non-zero replacements for x?

$$\frac{x^6}{x^4} = x^2$$

Notice the phrase "non-zero" used above. From your earlier courses in mathematics, you probably know that division by zero is not permitted. Therefore, let us agree to exclude 0 as a base when we consider any expression such as $\frac{x^6}{x^4}$. That is, the replacement set for x will be the non-zero numbers of arithmetic.

4. Do you agree that $\frac{y^3}{y^2} = y$ is a true sentence for all non-zero replacements for y? Try a few replacements. Notice, too, that the exponent "1" is consistent with the division pattern. That is, $\frac{y^3}{y^2} = y^{(3-2)}$, or y^1 (which we write simply as y).

5. What is the common name for $\frac{4}{4}$? for $\frac{18}{18}$? If n is any non-zero number of arithmetic, what is the quotient of n divided by n?

What is the quotient of $\frac{4^2}{4^2}$? What is the quotient of the square of any non-zero number divided by the square of the same number? the quotient of the cube of any non-zero number divided by the cube of the same number? of any power of any non-zero number divided by the same power of the same number?

Then do you agree that the quotient of each of the following is 1?

a. $\frac{7^2}{7^2}$ **b.** $\frac{15^6}{15^6}$ **c.** $\frac{9^4}{9^4}$ **d.** $\frac{x^3}{x^3}$ **e.** $\frac{y^7}{y^7}$

6. Suppose you had decided to subtract exponents in order to perform the division. Then, for part **a** in **5**, you might have written: $\frac{7^2}{7^2} = 7^{(2-2)}$. What does $2 - 2$ equal? But an exponent of "0" has no meaning as the word *exponent* has been used thus far. Our exponents have all been numbers from the set $\{1, 2, 3, \ldots\}$.

7. Suppose you had decided to subtract exponents to find the quotients in parts **b**, **c**, and **d** in **5**. You might have written: $\frac{15^6}{15^6} = 15^{(6-6)}$, or 15^0; $\frac{9^4}{9^4} = 9^{(4-4)}$, or 9^0; $\frac{x^3}{x^3} = x^{(3-3)}$, or x^0; and so on. Again, a "0" would be obtained in each case as an "exponent." But 0 as an exponent has no meaning thus far.

8. Let us see if we can establish a meaning for an exponent of "0." In each case in **5**, you found that the quotient of $\frac{7^2}{7^2}, \frac{15^6}{15^6}, \frac{9^4}{9^4}$, and so on, was 1. Then 7^0—if it is to be consistent with arithmetic—must be equal to 1; 15^0, by a similar reasoning, must be equal to 1; 9^0 must be equal to 1; x^0 must be equal to 1; and so on.

In view of this discussion, we make the following definition.

For any non-zero number a, $a^0 = 1$. (0^0 is not defined and is not considered to have any meaning.)

Testing the Zero Exponent

1. What does 6^0 equal? 6^3? What does $1 \cdot 216$ equal? Using exponents, $6^0 \cdot 6^3 = 6^{(0+3)}$, or 6^3. Are the results consistent?

2. What does 3^0 equal? 3^4? What does $\frac{81}{1}$ equal? Using exponents, $\frac{3^4}{3^0} = 3^{(4-0)}$, or 3^4. Are the results consistent?

3. Examine each of these tables carefully.

Powers of 10	*Powers of 2*
$10^5 = 100{,}000$	$2^5 = 32$
$10^4 = 10{,}000$	$2^4 = 16$
$10^3 = 1000$	$2^3 = 8$
$10^2 = 100$	$2^2 = 4$
$10^1 = 10$	$2^1 = 2$
$10^0 = 1$	$2^0 = 1$

In the table of powers of 10, notice that 10^4 is $\frac{1}{10}$ of 10^5; 10^3 is $\frac{1}{10}$ of 10^4; 10^2 is $\frac{1}{10}$ of 10^3; 10^1 is $\frac{1}{10}$ of 10^2. Is the relationship $10^0 = 1$ consistent with this pattern?

In the table of powers of 2, 2^4 is $\frac{1}{2}$ of 2^5; 2^3 is $\frac{1}{2}$ of 2^4; and so on. Is the relationship $2^0 = 1$ consistent with this pattern?

Have you found any instance where the 0 exponent causes any inconsistencies?

EXERCISES

1. Express the products in exponential form.

a. $8^4 \cdot 8^3$ **b.** $3^2 \cdot 3^6$ **c.** $9^5 \cdot 9^2$ **d.** $k^4 \cdot k^5$

e. $12^3 \cdot 12^3$ **f.** $10^8 \cdot 10^5$ **g.** $r^7 \cdot r^3$ **h.** $t^{10} \cdot t^{12}$

2. Express the quotients in exponential form.

a. $\frac{7^8}{7^2}$ **b.** $\frac{2^7}{2^4}$ **c.** $\frac{10^6}{10^5}$ **d.** $\frac{4^3}{4^3}$ **e.** $\frac{x^{11}}{x^9}$ **f.** $\frac{y^{15}}{y^5}$

3. What number is named by each of the following expressions?

a. $(5 \cdot 10^1) + (2 \cdot 10^0)$ **b.** $(4 \cdot 10^2) + (7 \cdot 10^1) + (8 \cdot 10^0)$

c. $(6 \cdot 10^2) + (9 \cdot 10^0)$ **d.** $(7 \cdot 10^2) + (1 \cdot 10^1) + (0 \cdot 10^0)$

Three Important Principles

As a result of the gradual widening of your work with exponents, you can now consider some useful general statements. Study your work on pages 32–36. Do you agree that the following is true?

If a is any non-zero number, and m and n are members of the set $\{0, 1, 2, 3, \ldots\}$:

I. $a^m \cdot a^n = a^{m+n}$

II. $\frac{a^m}{a^n} = a^{m-n}$, provided that $m \geq n$

III. $a^0 = 1$

These are important generalizations in mathematics, and you should be sure that you understand them. To test your understanding of these statements, evaluate a^{m+n} when a is replaced by 3, m by 4, and n by 2. Evaluate $\frac{a^m}{a^n}$ for these same replacements.

What does 7^0 equal? 5^0? What does $7^0 \cdot 5^0$ equal? $7 \cdot 5^0$?

You may wonder why we use the phrase "provided that $m \geq n$" in II. But it really does no more than state the conditions to which our investigation was limited. Later in the text, you will continue your work with exponents to carry your investigation farther.

EXERCISES

1. Evaluate the expression a^{m+n}:

a. when a is replaced by 2, m by 4, and n by 2

b. when a is replaced by 5, m by 3, and n by 1

c. when a is replaced by 8, m by 3, and n by 0

2. Evaluate a^{m-n}, using the replacements in Exercise **1.**

Proving a Number Relationship

1. Many interesting relationships may be discovered within the set of zero and the counting numbers. For example, complete the operations at the right and study the results.

$0 + 2$
$2 + 2$
$4 + 6$
$8 + 10$
$10 + 4$
$66 + 24$

Is each of the numbers you added an even number? Is the sum of each pair of even numbers in the table also an even number?

Do you think that the sum of *any* two even numbers is even?

2. In algebra, *thinking* that something is true is quite different from *proving* that it is true. Let us see how it might be proved that the sum of any two even numbers is an even number.

We begin by defining an even number as any member of the set $\{0, 1, 2, 3, \ldots\}$ that has 2 as a factor; that is, any number that is divisible by 2. Under this definition, is zero an even number? If x represents any member of the set $\{0, 1, 2, 3, \ldots\}$, does $2x$ represent an even number? Yes, because one of the factors of $2x$ is 2. If y represents any member of the set $\{0, 1, 2, 3, \ldots\}$, does $2y$ also represent an even number?

The expression $2x + 2y$ represents the sum of any two even numbers. By what mathematical principle may we rewrite this sum as $2(x + y)$?

Is the expression $2(x + y)$ the product of two expressions? Is 2 one of the factors of the expression $2(x + y)$? Then must $2(x + y)$ represent an even number, according to our definition of even numbers? If your answer is yes, you have proved what you set out to prove: The sum of any two even numbers is an even number.

3. Shown below is the way in which a mathematics student might write the proof in **2.**

By definition: An even number is any member of the set {0, 1, 2, 3, . . .} that has 2 as a factor.

Let the replacement set for x and y be {0, 1, 2, 3, . . .}.

Then $2x$ and $2y$ represent any two even numbers, and their sum may be represented as $2x + 2y$.

By the distributive principle: $2x + 2y = 2(x + y)$.

$2(x + y)$ represents the sum of any two even numbers, and 2 is a factor of $2(x + y)$. So the sum of any two even numbers is even.

4. To simplify the proof, one step was omitted. This is the step to show that the sum of any two members of {0, 1, 2, 3, . . .} is also a member of the set. Is the set {0, 1, 2, 3, . . .} closed under addition? Then does $x + y$ represent a member of the set?

The Sum of an Even Number and an Odd Number

1. Do you think that the sum of any even number and any odd number is always an odd number, or always an even number? Experiment with some pairs of numbers before you answer.

2. Let us see whether we can prove your conclusion to the question in **1.** We begin by defining what we mean by an odd number. Do you obtain an odd number when you add 1 to 0? when you add 1 to 4? when you add 1 to any even number? Then let us define an odd number as an element of the set {0, 1, 2, 3, . . .} that results from adding 1 to any even number.

If x represents any member of the set {0, 1, 2, 3, . . .}, does $2x$ represent an even number? If y represents any member of the set, does $2y$ represent an even number?

Both $2x$ and $2y$ represent even numbers, and we are concerned in our proof with both an odd number and an even number. Does $2y + 1$ represent an even number, or an odd number? Then we may use $2x$ to represent any even number and $2y + 1$ to represent any odd number.

The sum of $2x$ and $2y + 1$ may be written as $2x + (2y + 1)$. May we rewrite $2x + (2y + 1)$ as $(2x + 2y) + 1$? What principle permits us to do this?

May the expression $(2x + 2y)$ be rewritten as $2(x + y)$? Why? Then $2x + (2y + 1) = 2(x + y) + 1$.

You know from your work in the preceding section that $2(x + y)$ represents an even number. Then does $2(x + y) + 1$ represent an even number, or an odd number? Is the sum of any even number and any odd number always an odd number?

3. Study the following summary of the proof shown above.

By definition: An odd number is a number that is 1 more than an even number.

Let the replacement set for x and y be {0, 1, 2, 3, . . .}.

Then $2x$ represents an even number, and $2y + 1$ represents an odd number. Their sum may be represented as $2x + (2y + 1)$.

By the associative and distributive principles:

$$2x + (2y + 1) = 2(x + y) + 1.$$

$2(x + y) + 1$ represents a number that is 1 more than an even number. So the sum of any even number and any odd number is odd.

Another Proof Involving {0, 1, 2, 3, . . .}

In the preceding two sections, you found that number relationships could be proved by using sentences involving variables. Much of algebra, as you will see, involves work of a similar nature. Let us examine another relationship and see whether we can develop a proof for it.

1. Do you think that the sum of two odd numbers is always an odd number, or always an even number? Experiment with a few pairs of odd numbers to arrive at your answer.

2. Suppose you have concluded that the sum of any two odd numbers is an even number and are asked to prove that your conclusion is true. The first step would be to decide what expressions you may use to represent any two odd numbers. If x and y represent any members of the set {0, 1, 2, 3, . . .}, how would you represent any two odd numbers? (Remember that we have defined an odd number as a number that results from adding 1 to any even number.)

3. Your proof is to show that the sum of any two odd numbers is an even number. Using the expressions for any two odd numbers, write an expression that represents their sum. Do you obtain an expression of the form $(2x + 1) + (2y + 1)$?

4. You have learned that $2(x + y)$ represents an even number. You have also proved that the sum of two even numbers is an even number. Do you think an expression of the form $2(x + y) + 2$ will represent an even number? Explain your answer.

5. Using the expression you wrote for **3**, and using the results of the discussion in **4**, complete a proof of the fact that the sum of two odd numbers is an even number.

EXERCISES

1. a. Is the sum of three even numbers an even number, or an odd number? the sum of four even numbers? Give reasons for your answers.
b. Is the set of even numbers closed under addition?

2. a. Is the sum of three odd numbers an even number, or an odd number? Give reasons for your answer.
b. Is the set of odd numbers closed under addition?

3. If the variable in each of the following represents a member of the set $\{0, 1, 2, 3, \ldots\}$, tell which of these expressions definitely represents an even number, which definitely represents an odd number, and which will be an even number for some replacements.

a. $2y + 2$	**b.** $2x + 1$	**c.** $m + 2$
d. $q + 1$	**e.** $4y$	**f.** $(2m + 1) + (2n + 1)$
g. $4z + 1$	**h.** $1 + 2r$	**i.** $2 + r + r$
j. $d + 1 + d$	**k.** $(2y + 2z) + 2x$	**l.** $(2r + 2s) + (2z + 1)$

1. In mathematics, a system consists of elements and at least one ___?___. 2–3

2. Using a and b to represent numbers of arithmetic, write: 3–6
a. a generalization of the commutative principle of addition
b. a generalization of the commutative principle of multiplication

3. Using a, b, and c to represent numbers of arithmetic, write: 4–6
a. a generalization of the associative principle of addition
b. a generalization of the associative principle of multiplication

4. Are the counting numbers closed under addition? under multiplication? under division? 7–9

5. Using a, b, and c to represent numbers of arithmetic, write a generalization of the distributive principle. 16–17

6. Define sets A and C by description; sets B and D by tabulation. 18–19

$A = \{1, 3, 5, 7, 9\}$ $\quad B = \{\text{the months of the year}\}$
$C = \{101, 102, 103, \ldots\}$ $\quad D = \{\text{all counting numbers less than } 6\}$

7. The replacement set for the variable in each of the following sentences is the set $\{0, 1, 2, 3, \ldots\}$. Find the solution sets. 22–24

a. $p + 4 = 12$	**b.** $9q = 108$	**c.** $7x < 35$
d. $r + 15 = 15$	**e.** $y + 14 < 17$	**f.** $12k = 72$
g. $2z > 10$	**h.** $3s + 6 = 27$	**i.** $t + 5\frac{1}{2} = 10$

8. The replacement set for the variable in each of the following sentences is the set of numbers of arithmetic. Use set notation to show each solution set. 26–27

a. $r + 7 > 11$ **b.** $8s < 20$ **c.** $4t + 3 > 15$

9. On a number line, draw a graph of each of the solution sets that you found for the sentences in **8.** 27–29

10. Evaluate the following expressions at 5. 30–32

a. $y^3 + 25$ **b.** $2x^2 - 3x$ **c.** $4(9 + w^2)$ **d.** $\dfrac{t^2 - 4}{t + 2}$

11. Express the products in exponential form. 32–33

a. $10^2 \cdot 10^4$ **b.** $13^5 \cdot 13^3$ **c.** $r \cdot r^8$ **d.** $s^7 \cdot s^5$

12. Express the quotients in exponential form. 34–36

a. $\dfrac{6^6}{6^3}$ **b.** $\dfrac{8^9}{8^0}$ **c.** $\dfrac{11^7}{11^7}$ **d.** $\dfrac{h^6}{h^5}$ **e.** $\dfrac{k^{16}}{k^{11}}$

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 396
Challenge Page 411

2 The Integers and the Rational Numbers

A FURTHER LOOK AT SYSTEMS

The Identity Element In Chapter 1 you worked with the modulo 7 system and discovered that a study of a modular system can offer clues to characteristics of systems we use more frequently. Let us therefore continue our examination of modular arithmetic, using the modulo 7 system.

1. In each sentence below, find the replacement for the variable that will make the sentence true. (Refer to pages 9–10 if you need to refresh your memory about the operation $\oplus$ in modulo 7.)

a. $1 \oplus 0 = q$ **b.** $2 \oplus 0 = x$ **c.** $3 \oplus 0 = r$
d. $0 \oplus 4 = v$ **e.** $0 \oplus 5 = y$ **f.** $6 \oplus 0 = z$

Study the results of your work. Does it seem that when the operation $\oplus$ is applied to any number and zero we obtain identically that same number?

We say that the element 0 is the **identity element** for the operation $\oplus$ in the modulo 7 system. Since the operation $\oplus$ is similar to addition, let us agree to refer to this operation as addition in modulo 7. Then zero is the *identity element for addition in modulo 7.*

2. Consider the set of numbers consisting of zero and the counting numbers—that is, {0, 1, 2, 3, . . .}. Is there an identity element for the operation of addition? Name it.

3. It is important to understand that the identity element must be a member of the set in question. For instance, if you were asked whether there is an identity element for the operation of addition in the set of counting numbers, what would your answer be? Explain.

Inverse Elements

1. Consider the following sentence in modulo 7: $3 \oplus x = 0$. If we replace x by 4, will the sentence be true? (Draw a modulo 7 clockface and test your answer if you are not sure of it.)

2. Consider this sentence in modulo 7: $2 \oplus y = 0$. What replacement for y will make the sentence true?

3. In each of the following sentences, modulo 7, find the replacement that will make the sentence true.

a. $6 \oplus q = 0$ **b.** $r \oplus 5 = 0$
c. $s \oplus 4 = 0$ **d.** $0 \oplus z = 0$

4. Examine the pattern of the work in **1–3.** In each case, were we seeking a number such that when we added this number and a given number we would obtain the identity element as the result? List the various pairs of numbers that were found. For each number a in modulo 7, is there a number such that when we apply the operation $\oplus$ to this pair of numbers we obtain the identity element as the result?

If we apply an operation to a pair of elements and obtain the identity element as the result, the pair of elements are said to be **inverse elements** for that operation.

We say, therefore, that in modulo 7 such pairs as 6 and 1 or 5 and 2 are *inverse elements for addition,* or **additive inverses** of each other. What is the additive inverse of 4 in modulo 7? of 3? of 0?

5. Consider the set consisting of zero and all the counting numbers. Does the number 1 have an additive inverse in this set? That is, can you find a replacement for r that will make the following sentence true?

$$1 + r = 0$$

6. In the set of zero and the counting numbers, does 2 have an additive inverse? That is, can you find a replacement for s that will make the following sentence true?

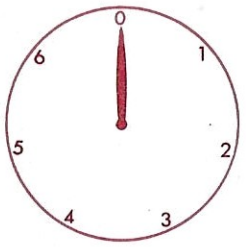
$$s + 2 = 0$$

Does 3 have an additive inverse? does 107?

Consider the number 0. Does it have an additive inverse within the set? That is, is there a replacement for y that will make the sentence $0 + y = 0$ a true statement? We say then that zero is its own additive inverse. In the set {0, 1, 2, 3, . . .}, is zero the only member that has an additive inverse within the set?

Inverse Operations in Modulo 7

The word *inverse* is probably not new to you. You may or may not be familiar with inverse elements. However, from your previous courses in mathematics, it is likely that you know that an operation is said to be the inverse, or opposite, of another operation when it "undoes" what the other operation has done. In our ordinary number system, for instance, what operation is the inverse of addition?



1. Let us now investigate an operation in modulo 7 which we will define as the inverse of addition in modulo 7. You remember that addition was defined as motion in a clockwise direction. Then might it be reasonable to define subtraction in modulo 7 as motion in a counterclockwise direction? Let us represent subtraction in modulo 7 by the symbol $\ominus$.

Since $3 \oplus 5 = 1$, should $1 \ominus 5$ equal 3? Starting at 1, count 5 units counterclockwise. At what number do you stop?

2. Since $3 \oplus 6 = 2$, what should $2 \ominus 6$ equal? $2 \ominus 3$? Check both of your answers, using the clockface.

3. Since $5 \oplus 5 = 3$, what should $3 \ominus 5$ equal? Check.

$\ominus$	0	1	2	3	4	5	6
0							
1						3	
2							
3							
4							
5							
6							

4. At the left is the form for a table in which the possible combinations for subtraction in modulo 7 are indicated. The numerals at the left represent the "minuends." The numerals at the top represent the "subtrahends." You can see that the entry in the second row represents the fact arrived at in **1** —namely, $1 \ominus 5 = 3$. Make a similar table and enter the results of your work for **2** and **3**.

5. Starting with 0 as a minuend, find the results for the various combinations—$0 \ominus 0$, $0 \ominus 1$, $0 \ominus 2$, and so on—and enter the results in your table.

6. Using 1 as the minuend, repeat the procedure described above. (That is, find $1 \ominus 0$, $1 \ominus 1$, $1 \ominus 2$, and so on.)

7. Look for a pattern in the entries you have made, and then complete the table. If you do not see the pattern, you may continue using the clockface to arrive at your answers.

8. Examine your table. Is subtraction in modulo 7 always possible? That is, for any two numbers a and b in modulo 7, does $a \ominus b$ name a member of the set? Another way to say this might be: Modulo 7 is $\underline{?}$ under subtraction.

9. Consider subtraction in the set of numbers consisting of zero and all the counting numbers—that is, the set $\{0, 1, 2, 3, \ldots\}$. Which, if any, member of the set is named by each of the expressions listed below?

a. $9 - 2$ **b.** $9 - 8$ **c.** $9 - 9$ **d.** $9 - 13$

Do you agree that the expression in **d** *does not* name a member of the set? Then is subtraction always possible in the set $\{0, 1, 2, 3, \ldots\}$?

10. Give another example that shows that subtraction is not always possible in the set $\{0, 1, 2, 3, \ldots\}$. We say that the set $\{0, 1, 2, 3, \ldots\}$ *is not* closed under subtraction.

Using the Identity Element and Additive Inverses

1. Suppose we wish to construct a set of numbers in which 0, 1, 2, 3, . . . are members and yet the operation of subtraction is always possible. Let us see how this might be done. For instance, let us consider the subtraction example $9 - 13$, which you found was not possible. We may write this example as:

$$9 - 13 = x$$

That is, we seek to construct a set of numbers in which $9 - 13$ will name a number (which we have represented by x).

Since subtraction is the inverse of addition, may we write this sentence as follows?

Then:
$$9 = x + 13$$
$$9 = x + (4 + 9)$$
$$9 = (x + 4) + 9$$

Explain each step. Now examine the last sentence in the sequence. Do you agree that for this sentence to be true the expression $x + 4$ must name the identity element, zero? And for $x + 4$ to name the identity element, will x have to be replaced by the additive inverse of 4?

2. Consider another example. Using a method similar to the one shown in **1**, we have the sequence at the right. Explain each step.

$$3 - 5 = y$$

For the last sentence in the sequence to be true, what number must be named by the expression $y + 2$? For $y + 2$ to name the identity element, y must be the additive inverse of what number?

$$3 = y + 5$$
$$3 = y + (2 + 3)$$
$$3 = (y + 2) + 3$$

EXERCISES

Using a method similar to that used in the preceding text discussion, determine what number would have to have an additive inverse in order for each of these subtraction examples to be possible.

1. $8 - 11$ **2.** $1 - 6$ **3.** $4 - 5$
4. $2 - 4$ **5.** $7 - 13$ **6.** $10 - 14$

On the basis of your work, do you agree that if we had a set of numbers in which 0, 1, 2, 3, . . . were members and in which each number had an additive inverse within the set, subtraction would always be possible?

THE INTEGERS

Constructing a New Set of Numbers

Let us construct a new set of numbers in which each member will have an additive inverse. We begin by agreeing that 0 is the identity element for addition. Then 0 will again be its own additive inverse; that is, $0 + 0 = 0$.

The additive inverse of 1 will be denoted by -1 (read "negative one"); the additive inverse of 2 will be denoted by -2 (read "negative two"); and so on. How will we represent the additive inverse of 3? of 4? of 8?

You will notice that a "$-$" sign is used with the numerals -1, -2, -3, and so on. However, this is *not* the "$-$" sign that indicates subtraction; this sign is a part of the numerals themselves.

1. By the definition of additive inverses, 1 is the additive inverse of -1. Why? Is 3 the additive inverse of -3? What is the additive inverse of -7? of -16? of 12? of -12?

For convenience, let us give the set of numbers $\{0, 1, -1, 2, -2, \ldots\}$ a name. Call it the set of **integers.** The numbers -1, -2, -3, . . . are called the **negative integers.** The numbers 1, 2, 3, . . . are called the **positive integers.** Zero is called the **zero integer** and is neither positive nor negative.

In the set of integers, which of the following sentences are true? Which are false? (Notice that parentheses are used when a negative numeral follows a sign of operation.)

	a	b	c
2.	$7 + (-7) = 0$	$9 + (-9) = 0$	$-3 + 3 = 0$
3.	$-1 + 1 = 0$	$4 + (-2 + 2) = 4$	$5 + (-4) = 0$

Explain why **3b** is true. Explain why you think **3c** is not true.

4. For each of the following sentences, find the replacement for the variable that will make the sentence a true statement. The replacement set is the set of integers.

a. $3 + x = 0$ **b.** $y + (-2) = 0$ **c.** $r + (-6 + 6) = 4$

EXERCISES

Tell which of the following sentences are true and which are false. If a sentence is neither true nor false, find the number that will make the sentence a true statement. The replacement set for each variable is the set of integers.

	a	b	c
1.	$0 = 6 + (-5)$	$11 + (-11) = 0$	$2 + (-2) = 1$
2.	$r + 1 = 0$	$15 + (-15) = 2 + (-2)$	$y + (-3) = 0$
3.	$12 + (-12) = 0$	$2 + [3 + (-3)] = 0$	$-9 + 9 = 0$
4.	$-12 + 0 = 12$	$0 + r = 0$	$8 + (-8) = q$

Interpreting Positive and Negative Numbers

In the preceding section, "-1" and "-2" were presented as though they were symbols that you had never seen before. It may be true that you are not familiar with these numerals as symbols for additive inverses of the positive integers 1, 2, 3, and so on. But most of you have seen similar symbols in everyday life. Try to give a few illustrations of their use. Then turn the page.

the Daily Globe

COLD WAVE UNBROKEN;
MERCURY AT -3°!

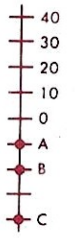
	LAMBS	TIGERS
Rushing yardage	275	10
Passing yardage	110	-45
Total yardage	385	-35

Look at the newspaper headline. Does the "—3°" mean 3 degrees below zero? If you see a headline that reads "Temperature hits 95°," does this mean a temperature below zero? If it does not mean that, what does it mean? What does a temperature of 3° indicate? a temperature of —7°?

Now look at the statistics for the football game. Does the positive integer 275 mean that the Lambs gained a total of 275 yards in ground plays? What does the 110 indicate? In the figures relating to the Tigers, what does the —45 indicate? What does the 10 indicate? (Try to explain the —35 that appears opposite "Total yardage.")

Do you agree that a loss in yardage and a gain in yardage represent opposing, or opposite, situations? What would a 0 in the "Rushing yardage" line represent?

Suppose we agree that in keeping track of individual plays we will use positive numbers for gains and negative numbers for losses. A gain of 5 yards followed by a loss of 5 yards can then be represented by $5 + (-5)$. What number would you use to represent this sum? How would you interpret this sum? Does this interpretation seem consistent with the idea of additive inverses that we used in constructing our new set of numbers?



EXERCISES

1. At the left is a vertical number scale in which the 0 point represents 0°F on a thermometer. The positive integers 10, 20, 30, 40, . . . correspond to temperature readings above zero. What temperature reading corresponds to the point labeled A? B? C?

2. After each 10-year census, the seats in the United States House of Representatives are redivided among the states, according to gain or loss in population. Below is a horizontal number scale used to compare gains or losses in seats that resulted from the 1960 census. The comparison is by regions (A, B, C, D) rather than by states. A positive integer (such as 5) is used to denote a gain in seats for a particular region; a negative integer (such as —5) denotes a loss in seats for a particular region.

The point labeled A is located one unit to the left of the 0 point. What integer would you use to identify this point? How would you interpret this "—1" in terms of the situation?

Point B corresponds to what integer? C? D? Interpret each of these in terms of the situation.

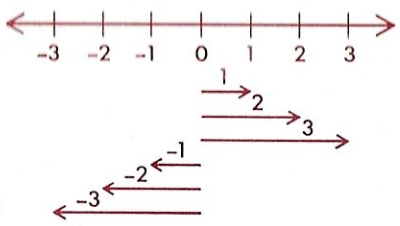
3. On the number scale shown above, is each point corresponding to a positive integer located to the right of the 0 point? Where is each point corresponding to a negative integer located?

The Number Line

Mathematically, a number line for the set of integers may be constructed as follows. On a line, we mark a point that is to correspond to the number 0. Then we choose a unit length and mark off successively, *to the right*, intervals the same length as the unit length. We then label the points corresponding to the positive integers 1, 2, 3, and so on.

Thus far, with 0 as a starting point, we have concerned ourselves with the portion of the line extending to the right of 0. You know, however, that a line extends indefinitely in both directions. So, beginning at 0, we now mark off successively, *to the left*, intervals one unit in length. We then label the points corresponding to —1, —2, —3, and so on. Below is a number line with points corresponding to the set of integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Where would you locate the point corresponding to —15? —27?

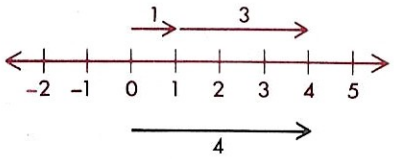


Addition on the Number Line

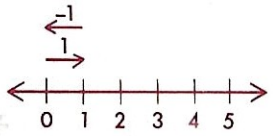
1. Study the diagram. For purposes of addition, the positive integer 1 can be associated with a line segment one unit in length, directed to the right. Similarly, 2 can be associated with a segment two units in length, directed to the right; 3 can be associated with a segment three units in length, directed to the right; and so on. The arrowheads indicate the direction of the segments.

In which direction does the segment associated with -1 extend? How many units in length is it? Describe the segment that is associated with -2. Draw a number line and sketch the line segment we can associate with -5; with 7; with -3. Indicate the direction in each case by an arrowhead.

2. The addition of 1 and 3 is easily pictured using directed line segments. (For convenience, let us refer to such segments as arrows.) Beginning at 0, we draw the arrow for 1; then, beginning at the head of the arrow representing 1, we draw an arrow of 3 units, directed to the right. The sum $1 + 3$ is represented by the arrow in black. That is, the sum $1 + 3$ is represented by an arrow that begins at 0 and ends at the point we reached in combining our two arrows.



Draw a number line and use arrows to represent $4 + 2$; $6 + 2$; $8 + 2$. In each case, is the second arrow 2 units in length? Does it begin at the head of the original arrow and extend to the right?

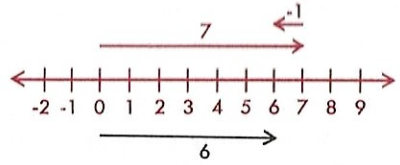


3. a. Consider now the addition of 1 and -1. We begin by drawing the arrow for 1. Then we draw the arrow for -1; namely, an arrow one unit in length, directed to the left.

Does the arrow for -1 take us back to 0? The sum, then, is an arrow of 0 length.

Is this consistent with our definition of 1 and -1 as additive inverses: $1 + (-1) = 0$?

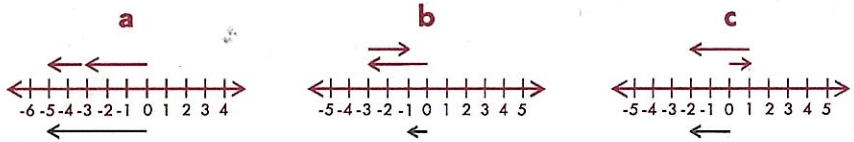
b. The illustration at the right represents the addition example $7 + (-1)$. The arrow in black represents the sum. Does $7 + (-1)$ seem to equal 6?



4. a. Represent the addition example $2 + (-2)$. Do you obtain an arrow of 0 length as the sum?

b. Represent the example $3 + (-2)$. What is the sum?

5. Below, three addition examples are represented. In each case, state the example and the sum.



EXERCISES

Use arrows to represent each of the following, and find the sums.

	a	b	c	d
1.	$4 + (-4)$	$5 + 3$	$-3 + 3$	$2 + (-6)$
2.	$6 + (-1)$	$4 + (-2)$	$-2 + (-4)$	$8 + (-1)$
3.	$3 + (-4)$	$5 + (-2)$	$-6 + 6$	$8 + (-3)$

Patterns in Addition

Draw a number line and represent each of the following by arrows. Then find the sum.

1.	**a.** $-3 + (-7)$	**b.** $-7 + (-3)$
2.	**a.** $-1 + (-6)$	**b.** $-6 + (-1)$
3.	**a.** $-5 + (-4)$	**b.** $-4 + (-5)$

Does it seem that the sum of two negative numbers is negative?

Use arrows to represent each of the following.

4.	**a.** $-2 + 7$	**b.** $7 + (-2)$
5.	**a.** $-3 + 6$	**b.** $6 + (-3)$
6.	**a.** $4 + (-5)$	**b.** $-5 + 4$

7. Examine your work in **1–6.** Does it appear that the addition of integers satisfies the commutative principle?

8. Find the number represented by $-3 + 6$. To this number, add -1. What is the sum? We can write this example as $(-3 + 6) + (-1)$.

Find the number represented by $6 + (-1)$. Add this number to -3. What sum do you obtain? We can write this example as $-3 + [6 + (-1)]$.

Using the procedure described above, find the sum for each of the following.

9. a. $[2 + (-4)] + 7$ **b.** $2 + [-4 + 7]$

10. a. $[-5 + 6] + (-3)$ **b.** $-5 + [6 + (-3)]$

Does it seem that the addition of integers satisfies the associative principle?

Find the sum represented by each of the following.

11. a. $7 + (-5)$ **b.** $-10 + 6$ **c.** $9 + (-7)$

12. a. $5 + (-6)$ **b.** $-13 + 8$ **c.** $-5 + 11$

In the examples in **11** and **12**, is it true that in each case one addend was a positive integer and the other a negative integer? Was the sum in each case a positive number? Was it in each case negative? Was it sometimes negative and sometimes positive?

Study the addends and the sums closely. Try to find a pattern to use in adding a positive number and a negative number.

13. If we assume that the addition of integers obeys the associative principle, we can find the sum of a positive integer and a negative integer quite easily without the number line. Consider, for instance, the addition of 7 and -5 (**11a**). We proceed as follows. Replacing 7 by $2 + 5$, we obtain:

$$\begin{aligned} 7 + (-5) &= (2 + 5) + (-5) \\ &= 2 + [5 + (-5)] \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

You can see that we have used the associative principle to obtain the second sentence. Explain why we can replace $5 + (-5)$ by 0. What property of zero makes the last step possible?

14. In a similar way, we can find the sum of -10 and 6 (**11b**). Here, however, we need to make use of $[-4 + (-6)]$ as another name for -10. Explain each of the steps shown below.

$$\begin{aligned} -10 + 6 &= [-4 + (-6)] + 6 \\ &= -4 + (-6 + 6) \\ &= -4 + 0, \text{ or } -4 \end{aligned}$$

EXERCISES

Find the sum for each of the following examples. In Exercises **1** and **2**, use the method of illustrative examples **13** and **14** on page 52.

	a	b	c	d
1.	$12 + (-9)$	$-7 + 2$	$8 + (-11)$	$9 + (-4)$
2.	$-3 + 9$	$-27 + 15$	$81 + (-76)$	$-111 + 36$
3.	$16 + (-7)$	$16 + (-8)$	$16 + (-9)$	$16 + (-11)$
4.	$3 + (-18)$	$5 + (-18)$	$7 + (-18)$	$15 + (-18)$

Order of the Integers

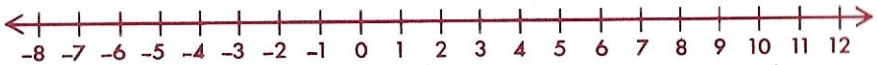
You will remember from Chapter 1 that the symbol "$<$" is read "is less than." Is the sentence $8 < 11$ true? the sentence $5 < 9$? Write a sentence comparing 3 and 1, using the symbol "$<$."

The symbol "$<$" is called a symbol of **order**, and when we use this symbol in a sentence such as $8 < 11$, we say that we have *ordered* the numbers 8 and 11.

1. If you are asked to *order* two positive integers or a positive integer and zero, you have no difficulty in doing it. For example, order the following pairs of numbers, using the symbol "$<$."

a. 8, 0 **b.** 2, 1 **c.** 0, 1

2. The next question is: How shall we establish order for all the integers? Look at the number line below. Is the point corresponding to 8 to the left of the point corresponding to 11? Is 5 to the left



of 9? Is 1 to the left of 3? 0 to the left of 1? For the set of numbers $\{0, 1, 2, 3, \ldots\}$, do you agree that if $a < b$ the point corresponding to a lies to the left of the point corresponding to b?

Suppose we agree to preserve this pattern for the set of integers. Then, since the point corresponding to -1 is to the left of 0, we say that $-1 < 0$. Is it true, then, that $-3 < -1$? $-5 < -3$? $-4 < 1$? $-1 < 1$?

Is it true that $0 < -4$? $-1 < -2$?

3. It is helpful to use a number line to determine which of two numbers is less than another. But let us now try to form a definition for "is less than" that does not use the number line.

For each of the following sentences, find the replacement for the variable that will make the sentence true.

a. $8 + q = 11$ **b.** $-1 + r = 0$
c. $-3 + s = -1$ **d.** $-5 + t = -3$
e. $-4 + u = 1$ **f.** $-1 + v = 1$

Did you find that in each case a positive number had to be added to the first number to obtain the second? This is the basis on which we form our definition of "is less than."

▶ A number a is less than a number b if a positive number must be added to a to obtain b.

Did you notice in **3a–f** that the numbers used were taken from the pairs given in **2**, page 53? Using the left-to-right pattern, you decided that $-3 < -1$, $-5 < -3$, and so on. But now you have a precise definition of what you mean by "is less than." Then -3 is less than -1, because 2 must be added to -3 to obtain -1.

4. Consider the sentences (from **2**) that you decided were false: $0 < -4$; $-1 < -2$. To see why they are false (according to the definition), we merely write the following sentences and find the replacement that makes the sentence true.

a. $0 + m = -4$ **b.** $-1 + n = -2$

Does -4 make sentence **a** true? Does -1 make sentence **b** true? Then the definition of "is less than" is not satisfied.

5. Show that $-4 < 0$ by solving the sentence $-4 + x = 0$. In a similar way, show that $-2 < -1$.

6. Is each negative integer less than each positive integer? Is each negative integer less than zero?

7. If an integer a is less than an integer b, does this mean that b is greater than a? In symbols, we can say that if $a < b$, then $b > a$. (The symbol ">," you remember, means "is greater than.")

8. Using "is greater than," rephrase the relationships you discovered in **6.**

9. What is another name for each of the following sets of numbers?
a. $\{y \mid y \text{ is an integer less than zero}\}$
b. $\{x \mid x \text{ is an integer greater than zero}\}$

EXERCISES

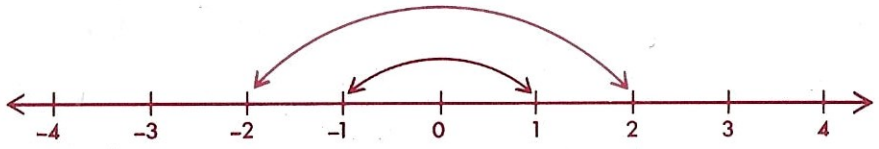
Using the symbol "<," order the numbers in each of the following. Then write a sentence (as in **5**, page 54) showing what number must be added to the lesser number to obtain the greater number.

1. a. $-1, 1$ **b.** $1, -5$ **c.** $2, -3$ **d.** $-3, -4$
2. a. $-17, 3$ **b.** $-15, -20$ **c.** $6, -7$ **d.** $0, -9$

Opposites and Absolute Value

The additive inverse of a number is also known as the **opposite** of the number. By studying the number line below, we can interpret the word *opposite*. Are the points corresponding to -1 and 1 on opposite sides of the 0 point? Are they the same distance from 0?

Is this true of 2 and -2? of -3 and 3?



▶ For any number x, the opposite of x is designated as $-x$.

1. If x is a positive integer, $-x$ is a negative integer. If x is a negative integer, the opposite of x is a positive integer. That is, if x represents the number -2, then $-x$ (the opposite of x) is 2.

Therefore, when you see an expression such as $-x$, remember that it does not necessarily indicate a negative number. It indicates the *opposite* of x, and we do not know whether $-x$ is negative or positive until we make a replacement for x.

What is $-x$ if x is replaced by 7? by -7? by -12?

2. In mathematics, we frequently make use of the idea of the **absolute value** of a number. The symbol for the absolute value of a number x is "$|x|$" and we define absolute value as follows:

▶ The absolute value of a positive number is the number itself.
That is, $|6| = 6$; $|11| = 11$; and so on.
The absolute value of a negative number is its opposite.
That is, $|-6| = 6$; $|-11| = 11$; and so on.
The absolute value of 0 is 0. That is, $|0| = 0$.

The sentence $|6| = 6$ is read "The absolute value of 6 is 6," and the sentence $|-6| = 6$ is read "The absolute value of negative 6 is 6." How would you read $|-11| = 11$? $|212| = 212$?

3. From the definition and from the illustrations, you can see that the absolute value of any integer is always a non-negative integer. What does $|-275|$ equal? $|275|$?

Using mathematical symbolism, we can define the absolute value of a number as follows:

$|x| = x$ if x is a positive number or zero
$|x| = -x$ if x is a negative number

4. In each of the following, find the replacement for the question mark that makes the sentence a true statement.

a. $|8| = ?$ **b.** $|-8| = ?$ **c.** $|-9| = ?$
d. $|-14| = ?$ **e.** $|14| = ?$ **f.** $|9| = ?$

5. Suppose you are given the open sentence $|x| = 3$ where the replacement set is the set of all integers. If you replace x by -3, will you obtain a true statement? Is there any other number that will yield a true statement? If so, what is it? Then $\{-3, 3\}$ is the solution set of $|x| = 3$. What is the solution set of $|y| = 7$? of $|z| = 0$?

6. Suppose you are asked to find the solution set of the sentence $|r| = -7$. Can you find a replacement for r that will make the sentence true? Explain your answer. Then the solution set of $|r| = -7$ is $\emptyset$, the empty set.

7. a. Suppose you are asked to find what $|6| + |-7|$ equals. What is the absolute value of 6? What is the absolute value of -7? Then what does $|6| + |-7|$ equal?

b. On the other hand, suppose you are asked to determine what $|6 + (-7)|$ equals. Here you are being asked to find the absolute value of an expression that is an indicated sum. What integer is named by the expression $6 + (-7)$? What does $|-1|$ equal? Then $|6 + (-7)| = 1$.

c. Does $|6| + |-7|$ equal $|6 + (-7)|$?

8. What does $|-7| + |-9|$ equal? What does $|-7 + (-9)|$ equal? Does $|-7| + |-9|$ equal $|-7 + (-9)|$?

EXERCISES

In each of the following, find the replacement for the question mark that makes the sentence a true statement.

	a	b	c
1.	$\|-21\| = ?$	$\|33\| = ?$	$\|-10\| = ?$
2.	$\|147\| = ?$	$\|-52\| = ?$	$\|21\| = ?$
3.	$\|-901\| = ?$	$\|555\| = ?$	$\|-33\| = ?$

4. If the replacement set is the set of integers, what is the solution set of $|n| = 15$? $|m| = -7$? $|r| = 50$?

Find what each of the following equals.

5. $|3| + |-4|$ **6.** $|-5| + |-7|$ **7.** $|-8| + |8|$
8. $|9| + |9|$ **9.** $|-17| + |7|$ **10.** $|-6| + |-4|$

Using Absolute Value

The language of absolute value can be helpful in speaking about the sum of two integers.

1. a. You know that the sum of two positive integers is always a positive integer. Is the absolute value of the sum equal to the sum of the absolute values of the two numbers? That is, does $|6 + 5|$ equal $|6| + |5|$?

b. You know also that the sum of two negative integers is always a negative integer. Is the absolute value of the sum equal to the sum of the absolute values of the two numbers? Give three examples to support your answer.

2. Is the sum of a positive integer and a negative integer always a positive integer? always a negative integer? Find the sums.

a. $-4 + 11 = ?$ **b.** $-8 + 6 = ?$

In part **a**, is the sum positive, or is it negative? Which is greater, the absolute value of *positive 11* or the absolute value of *negative 4*? Is the absolute value of the sum equal to $|-4| + |11|$? Is the absolute value of the sum equal to $|11| - |-4|$?

In part **b**, is the sum positive, or is it negative? Which is greater, the absolute value of *negative 8* or the absolute value of *positive 6*? Is the absolute value of the sum equal to $|-8| + |6|$? Is the absolute value of the sum equal to $|-8| - |6|$?

3. If we are adding a positive integer and a negative integer and the positive integer has the greater absolute value, will the sum be positive? If the negative integer has the greater absolute value, will the sum be negative?

$-17 + 26 = ?$

$17 + (-26) = ?$

Below is a general statement of the results of this investigation.

If a is a positive integer and b is a negative integer:
the sum $a + b$ is positive if $|a| > |b|$;
the sum $a + b$ is negative if $|b| > |a|$.
To find the absolute value of the sum, we subtract the lesser absolute value from the greater absolute value.

4. Test your understanding of the general statement above by telling at a glance whether the sum of each of the following is positive or negative. Then write the sum.

a. $-111 + 61$ **b.** $-45 + 15$ **c.** $27 + (-57)$
d. $95 + (-32)$ **e.** $-112 + 107$ **f.** $-81 + 106$

5. Suppose x is a positive integer and y is a negative integer. What number do we obtain for the sum of x and y if $|x| = |y|$? (If a positive integer and a negative integer have the same absolute value, what relationship exists between these numbers?)

6. Think back over your work in the addition of integers. In each case, when you added two integers, did you obtain an integer as the result? Do you agree that the set of integers is closed under addition?

EXERCISES

Find the sums.

	a	b	c
1.	$43 + (-13)$	$-23 + 27$	$-43 + 13$
2.	$-16 + 72$	$124 + (-37)$	$55 + (-66)$
3.	$-53 + 44$	$96 + (-99)$	$512 + (-231)$
4.	$-611 + 611$	$-86 + 34$	$-72 + 127$
5.	$357 + (-412)$	$329 + (-298)$	$-400 + 318$
6.	$863 + (-638)$	$-722 + 277$	$945 + (-945)$

Subtraction of Integers

You know that the operation of subtraction is the inverse of addition. Throughout your earlier work in arithmetic and mathematics, you have always understood (perhaps without even thinking very often about it) that the sentence $26 - 10 = 16$ is true *because* $10 + 16 = 26$. That is, when we wish to determine the difference between two numbers c and b (in symbols, $c - b$), we seek a number a such that the sum of b and a will equal c. We can write:

$$c - b = a \quad \text{if and only if} \quad b + a = c$$

1. To find the difference $7 - (-1)$, we can let x represent this difference and write:

$$7 - (-1) = x$$

Then we can write: $(-1) + x = 7$

What replacement for x will make the sentence true? Then the difference $7 - (-1)$ is 8.

2. To find the difference $2 - 5$, you can write: $2 - 5 = y$. Then:

$$5 + y = 2$$

What replacement for y will make the sentence true? Then the difference $2 - 5$ is -3.

3. For each of the following, find the difference. Use the method shown in **1** and **2**. (Retain your results for later use.)

a. $8 - (-2)$ **b.** $-1 - 7$ **c.** $0 - 5$
d. $-3 - (-4)$ **e.** $-5 - 5$ **f.** $3 - (-9)$

In each of the following pairs of examples (**a** and **b**), one involves subtraction, the other addition. In each case, first find the difference indicated in part **a**, then find the sum indicated in part **b**.

4.	**a.** $7 - 2$	**b.** $7 + (-2)$
5.	**a.** $8 - 9$	**b.** $8 + (-9)$
6.	**a.** $5 - 10$	**b.** $5 + (-10)$
7.	**a.** $4 - (-1)$	**b.** $4 + 1$
8.	**a.** $11 - (-3)$	**b.** $11 + 3$
9.	**a.** $-1 - 8$	**b.** $-1 + (-8)$
10.	**a.** $3 - 12$	**b.** $3 + (-12)$

Study the results in **4–10**. In each case, was the sum you found in part **b** equal to the difference in part **a**?

Are the following sentences true: $7 - 2 = 7 + (-2)$; $8 - 9 = 8 + (-9)$; $5 - 10 = 5 + (-10)$; $4 - (-1) = 4 + 1$? Does it appear that the difference in any subtraction example is equal to the sum of the minuend and the opposite of the subtrahend? We have not proved this relationship, but let us agree to accept it without proof and to define subtraction in terms of addition. We state the agreement as follows.

For any integers c and b, the difference $c - b = c + (-b)$.

In other words, subtracting a number is the same as adding its opposite, or additive inverse.

11. Using the definition we have just established, find the differences in the examples in **3**, page 59. Check your results with the results you obtained earlier.

12. Find the differences: $8 - (-4)$; $-1 - 1$; $0 - 3$; $0 - (-4)$; $11 - 19$.

13. For any integer x, is it true, or is it false, that $0 - x = -x$? Show your reasoning, using mathematical sentences.

EXERCISES

Find the differences.

	a	b	c
1.	$-36 - 12$	$-11 - (-9)$	$62 - (-52)$
2.	$147 - 43$	$-17 - 71$	$-84 - (-133)$
3.	$19 - (-19)$	$57 - (-62)$	$38 - 83$
4.	$-48 - 24$	$-110 - (-15)$	$-50 - 50$
5.	$90 - (-5)$	$123 - 25$	$-101 - 111$
6.	$-27 - (-53)$	$50 - 50$	$21 - (-123)$
7.	$52 - 162$	$-55 - 45$	$-61 - (-49)$
8.	$-15 - 15$	$75 - (-25)$	$92 - 16$
9.	$-23 - 28$	$-39 - (-93)$	$50 - (-50)$
10.	$9 - (-18)$	$-57 - (-57)$	$24 - 48$

Perform the indicated operation in each case.

11.	$-82 + (-75)$	$-29 - 23$	$69 + (-27)$
12.	$-25 - (-15)$	$-44 + 28$	$78 - (-72)$
13.	$-235 + (-165)$	$89 - 137$	$222 + (-167)$
14.	$-64 - 46$	$-172 + 158$	$-128 - (-44)$
15.	$546 + 645$	$142 - (-42)$	$-172 + 258$
16.	$234 - 432$	$-69 + (-74)$	$-58 - 118$

Commutative Group

What have we discovered about the integers under the operation of addition?

(1) For any two integers x and y, do you agree that $x + y$ is a member of the set of integers? We say that the set of integers is closed under the operation of addition.

(2) Is addition of integers commutative? In symbols, for any two integers x and y:

$$x + y = y + x$$

(3) Is addition of integers associative? In symbols:

$$(x + y) + z = x + (y + z)$$

(4) The set of integers has an identity element for addition. What is it? For any integer x:

$$x + 0 = 0 + x = x$$

(5) For any integer x, there is an integer $-x$ (the opposite of x) such that x and $-x$ are inverse elements for addition. That is, $x + (-x)$ equals 0, the identity element for addition.

A set of elements that has these properties with respect to a given operation—that is, a set that is closed under the operation, obeys the commutative and associative principles, and has an identity element and inverse elements for the given operation—is called a **commutative group.** Thus, the set of integers is a commutative group under the operation of addition.

In Chapter 1 we spoke of the fact that one basic study in algebra was the structure of systems. One of the most important structures is that of the commutative group. So far we have discussed only the idea of a commutative group under the operation of addition. We will soon consider the idea of a commutative group under another operation.

Meanwhile you can, on your own, consider these questions:

1. Is the set $\{1, 2, 3, \ldots\}$ a commutative group under addition? If not, why not? Which property or properties are not satisfied?

2. Is the set $\{0, 1, 2, 3, \ldots\}$ a commutative group under addition? If not, why not?

3. Does the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ under the operation $\oplus$, modulo 7, represent a commutative group? Explain your answer, using the various properties.

Summary

Since the idea of a commutative group is important throughout our work, the following generalization is given for your reference.

Given: A set of elements $G = \{a, b, c, \ldots\}$ and an operation which we will denote by "$*$" that applies to two elements at a time. G is a commutative group if it has the following properties.

1. For any two members of G, $a * b$ is a member of G.
   (Closure property)
2. For any two members of G, $a * b = b * a$.
   (The operation is commutative.)
3. For any three members of G, $(a * b) * c = a * (b * c)$
   (The operation is associative.)
4. There exists a member of G—call it e—such that for each member a of G, $a * e = e * a = a$.
   (Existence of an identity element)
5. For each member of G, there exists a member of G such that, applying the operation to this pair of elements, we obtain the identity element. Designating the pairs as a and a_i, b and b_i, and so on, we have: $a * a_i = e$; $b * b_i = e$; and so on.
   (Every element in G has an inverse element in G.)

EXERCISES

1. Is the set of odd numbers $\{1, 3, 5, \ldots\}$ a commutative group under addition? Explain your answer by telling which, if any, of the various properties are not satisfied.

2. Does the set of numbers $\{0, 1, 2\}$ under the operation $\oplus$, modulo 3, represent a commutative group? Explain your answer fully.

Multiplication of Integers

Let us now investigate and interpret the operation of multiplication as it relates to integers. We want to define this operation so that multiplication of integers will obey the same principles as multiplication in the set $\{0, 1, 2, 3, \ldots\}$. That is, we want multiplication of integers to be commutative, associative, and distributive over addition. There are other properties that are important, and these we will examine in **1** and **2** on page 63.

1. In the set $\{0, 1, 2, 3, \ldots\}$, is there an *identity element for the operation of multiplication*? That is, is there an element—call it e—such that when we find the product of e and any number in the set, we obtain that number as the result? To answer this question, we must find a replacement for e that will make all of the following sentences true. What number will do this?

$$e \cdot 6 = 6 \qquad 9 \cdot e = 9 \qquad 111 \cdot e = 111 \qquad e \cdot 74 = 74$$

Is the number 1 the identity element for multiplication in the set $\{0, 1, 2, 3, \ldots\}$?

Let us agree to maintain this property in the multiplication of integers. That is, $1 \cdot (-7) = -7 \cdot 1 = -7$; $1 \cdot (-9) = -9 \cdot 1 = -9$; and so on.

2. In the set $\{0, 1, 2, 3, \ldots\}$, what is the product of $0 \cdot 7$? $9 \cdot 0$? $0 \cdot 15$? Does $n \cdot 0$ equal 0 for any replacement for n?

As you can see, the number 0 has a special property in multiplication. We may call it the **multiplication property of zero** and state it as follows for the set $\{0, 1, 2, 3, \ldots\}$:

The product of any number and zero equals zero.
In symbols, $0 \cdot a = a \cdot 0 = 0$ for all replacements for a.

Let us agree that this, too, is a property we will maintain for multiplication of integers. That is, $0 \cdot (-6) = 0$, $-121 \cdot 0 = 0$, and so on. What is $0 \cdot 0$? $0 \cdot 1$? $0 \cdot (-1)$?

3. a. As you know, a multiplication example such as $3 \cdot 4$ may be thought of as successive addition. That is, $3 \cdot 4 = 4 + 4 + 4$, or 12. Represent by successive addition: $2 \cdot 11$.

b. Let us consider the multiplication of a negative integer by a positive integer: $3 \cdot (-4)$. Using the idea of successive addition, we can write:

$$3 \cdot (-4) = (-4) + (-4) + (-4)$$

What is the product?

4. Find the products, using the idea of successive addition.

a. $2 \cdot (-11)$	**b.** $7 \cdot (-9)$	**c.** $5 \cdot (-5)$
d. $6 \cdot (-12)$	**e.** $5 \cdot (-14)$	**f.** $11 \cdot (-13)$

Examine your work in **4a–f.** If a negative integer is multiplied by a positive integer, is the product positive, or is it negative?

5. Consider next the multiplication of a positive integer by a negative integer: $-4 \cdot 3$. The idea of successive addition is not appropriate here. However, we can make use of the fact that we want multiplication to be commutative. That is, we want $-4 \cdot 3$ to equal $3 \cdot (-4)$. Look back at **3b**, page 63. What must $-4 \cdot 3$ equal if multiplication is to be commutative?

6. If multiplication is to be commutative, what must $-11 \cdot 2$ equal? $-9 \cdot 7$? $-5 \cdot 5$? $-12 \cdot 6$?

7. Find the products.

a. $-11 \cdot 12$ **b.** $-8 \cdot 15$ **c.** $-6 \cdot 22$
d. $-18 \cdot 20$ **e.** $-21 \cdot 43$ **f.** $-76 \cdot 82$

Examine your work in **5–7.** If a positive integer is multiplied by a negative integer, is the product positive, or is it negative?

Is this consistent with the relationships $-7 \cdot 1 = -7$, $-9 \cdot 1 = -9$, and so on, established in **1**, page 63?

8. There remains just one case of multiplication of integers to consider: the multiplication of a negative integer by another negative integer. Consider, for example, $-5 \cdot (-6)$. Two possibilities seem obvious: Either the product is 30 or it is -30.

To determine which number we will use as the product, we proceed in a somewhat roundabout way. The "roundabout" way makes use of the multiplication property of zero and the fact that we want multiplication to be distributive over addition.

We begin by writing sentence (1) at the right. Why is this sentence true?

(1) $-5 \cdot 0 = 0$

Next we replace the "0" of "$-5 \cdot 0$" by the indicated expression. Why can we do this?

$\uparrow$

$[6 + (-6)]$

We now have sentence (2). Next, we make use of the fact that multiplication is to be distributive over addition. We obtain, then, sentence (3).

(2) $-5 \cdot [6 + (-6)] = 0$

(3) $(-5 \cdot 6) + [-5 \cdot (-6)] = 0$

We know that $-5 \cdot 6$ is -30. We can therefore obtain sentence (4). You can see that the expression in brackets represents the product in which we are interested. Must the product be 30 for the sentence to be true? Why?

(4) $-30 + [-5 \cdot (-6)] = 0$

Let us review what we have done. We have found that if multiplication of integers is constructed with the characteristics established on pages 63–64, **1–7**, and if we want multiplication to be distributive over addition, then $-5 \cdot (-6)$ must equal 30.

9. In a similar way, we can find what the product of any two negative integers must equal. Suppose, for instance, we are interested in the product $-4 \cdot (-7)$. We proceed as shown at the right.

$-4 \cdot 0 = 0$

$-4 \cdot [7 + (-7)] = 0$

$-28 + [-4 \cdot (-7)] = 0$

Explain why each of the sentences is true.

Must $-4 \cdot (-7)$ equal 28? Why?

10. Do you think that if we want the multiplication property of zero and the distributive principle to hold true, the product of two negative integers will always be a positive integer? Find the following indicated products by the procedure shown in **8** and **9**, if you are not sure. If you are reasonably certain that the pattern will hold, merely state the products.

a. $-5 \cdot (-7)$ **b.** $-9 \cdot (-11)$ **c.** $-6 \cdot (-25)$
d. $-4 \cdot (-32)$ **e.** $-7 \cdot (-18)$ **f.** $-8 \cdot (-17)$

On the basis of your work with multiplication of integers, do you agree with the following generalizations?

If a and b are both positive numbers, or both negative numbers, their product is positive.

If one of the factors (either a or b) is a negative number and the other is a positive number, their product is negative.

11. Is the product of two integers always an integer? Do you agree that the set of integers is closed under multiplication?

12. On page 62, we said that we wanted multiplication in the set of integers to be associative. Yet we have not had to make use of associativity or to consider it thus far. Let us now investigate whether any decisions we have made have "destroyed" the principle of associativity. Determine whether the following sentences are true.

a. $[2 \cdot 3] \cdot (-4) = 2 \cdot [3 \cdot (-4)]$
b. $[5 \cdot (-2)] \cdot 3 = 5 \cdot [-2 \cdot 3]$
c. $[-4 \cdot (-4)] \cdot 5 = -4 \cdot [-4 \cdot 5]$
d. $[7 \cdot (-1)] \cdot (-2) = 7 \cdot [-1 \cdot (-2)]$

Does it seem that the associative principle holds true for multiplication in the set of integers?

EXERCISES

Find the products.

	a	b	c
1.	$8 \cdot (-17)$	$-5 \cdot 57$	$-3 \cdot (-93)$
2.	$-25 \cdot 12$	$-7 \cdot (-77)$	$14 \cdot (-10)$
3.	$-2 \cdot (-88)$	$15 \cdot (-19)$	$-13 \cdot 13$
4.	$-8 \cdot (-20)$	$34 \cdot (-7)$	$-12 \cdot (-13)$
5.	$5 \cdot (-66)$	$12 \cdot 44$	$-15 \cdot (-10)$
6.	$-8 \cdot 43$	$-1 \cdot (-1)$	$33 \cdot (-22)$
7.	$-4 \cdot 23$	$75 \cdot 11$	$-68 \cdot (-4)$
8.	$6 \cdot (-87)$	$-85 \cdot (-21)$	$-30 \cdot 17$
9.	$-72 \cdot (-15)$	$-16 \cdot 61$	$9 \cdot 128$
10.	$-39 \cdot 3$	$12 \cdot (-15)$	$-18 \cdot (-30)$

Division of Integers

You know that the operation of division is the inverse of multiplication. The sentence $28 \div 7 = 4$ is true *because* $7 \cdot 4 = 28$. Or, using algebraic symbolism, we say:

$$c \div b = a \qquad \text{if and only if} \qquad b \cdot a = c$$

1. What does $-7 \cdot (-4)$ equal? Then what does $28 \div (-7)$ equal?
2. What does $-7 \cdot 4$ equal? Then what does $-28 \div (-7)$ equal?
3. What does $7 \cdot (-4)$ equal? Then what does $-28 \div 7$ equal?
4. Suppose you are asked to find the quotient for the example $-63 \div 9$. You can let y represent the quotient and write:

$$-63 \div 9 = y$$

Then you can write:

$$9 \cdot y = -63$$

What replacement for y will make the sentence a true statement? Then -63 divided by 9 is -7.

5. Frequently, your work in division of integers can be done mentally. To find the quotient $48 \div (-8)$, you can think: "What number multiplied by -8 equals 48?" Is the quotient 6, or is it -6?

6. Find the quotient in each of the following. Use the fact that the operation of division is the inverse of multiplication.

a.	$15 \div (-3)$	b. $-15 \div 3$	c. $-15 \div (-3)$
d.	$64 \div (-16)$	e. $-64 \div 8$	f. $-64 \div (-32)$
g.	$96 \div (-48)$	h. $-81 \div 27$	i. $-66 \div (-11)$

On the basis of your experience with multiplication of integers and your work in **1–6**, answer the following questions.

When a positive number is divided by a negative number, is the quotient positive, or is it negative?

When a negative number is divided by a positive number, is the quotient positive, or is it negative?

When a negative number is divided by a negative number, is the quotient positive, or is it negative?

Uniqueness

If you are asked to find the sum of two numbers such as 7 and 3, do you agree that there is *just one number* that is the sum of 7 and 3? We can name this number in many ways—for instance, as "10" or as "$(3 + 4) + 3$" or as "$5 + (2 + 3)$"—but there is only one number that is the sum. We say that the sum of 7 and 3 is **unique.** Is the sum of 8 and -2 unique? Do you agree that the sum of any two integers is unique?

Is the product of two integers such as 8 and -2 unique? What is this product? Do you agree that the product of any two integers is unique?

Is the difference $8 - (-2)$ unique? Do you agree that the difference between any two integers is unique?

Is the quotient of two integers such as $8 \div (-2)$ unique?

In mathematics, it is important to have a unique result for an operation involving a pair of numbers. Do you see that our ordinary arithmetic calculations would not be useful to us if products, sums, differences, and quotients were not unique?

More about Zero

Let us investigate the use of zero as a divisor. You know that $a \div b = c$ if and only if $b \cdot c = a$. Using this fact, let us consider the two possibilities: the dividend (a) is a number other than zero; the dividend is itself zero.

Case 1: $a \neq 0$. We are given that $a \div 0 = c$. Since division is the inverse of multiplication, we know that:

$$\text{if } a \div 0 = c, \text{ then } 0 \cdot c = a$$

Examine the sentence $0 \cdot c = a$. No matter what number c is replaced by, will $0 \cdot c$ equal 0? Why? But we agreed that $a \neq 0$. Therefore, $0 \cdot c$ cannot equal a. Thus, we cannot define division by zero when the dividend is a non-zero number.

Case 2. $a = 0$. Now we start with $0 \div 0 = c$. Again we make use of the fact that division is the inverse of multiplication.

$$\text{if } 0 \div 0 = c, \text{ then } 0 \cdot c = 0$$

This sounds all right. We have not violated the definition of division, since the product of any number and zero is zero. But what replacement shall we use for c? Could we use 1 and obtain a true sentence? Could we use -18? Then is the quotient unique? Since we want division to be unique, can we define division by zero when the dividend is zero?

Then division by zero is not defined. From this point on, we must be careful when we write "$a \div b$" to make it clear that b cannot be replaced by zero. We write:

$$a \div b = c, \quad b \neq 0$$

Division of Integers: Closure

Excluding zero as a divisor, do you think that in the set of integers division is always possible?

1. Consider the sentence $1 \div 2 = n$ where the replacement set for n is the set of integers. Is there a replacement that will make the sentence a true statement?

2. In each of the sentences below, the replacement set for the variable is the set of integers. In each case, try to find a replacement that will make the sentence a true statement.

a. $3 \div 4 = x$ **c.** $8 \div (-2) = r$
b. $-12 \div 4 = y$ **d.** $6 \div (-5) = z$

Is the set of integers closed under the operation of division?

EXERCISES

Find the quotients.

	a	b	c
1.	$-45 \div 5$	$55 \div (-11)$	$-120 \div (-15)$
2.	$84 \div (-7)$	$-96 \div (-3)$	$-52 \div 13$
3.	$-174 \div (-29)$	$-152 \div 19$	$92 \div (-2)$
4.	$36 \div 12$	$81 \div (-9)$	$-104 \div 8$
5.	$90 \div (-10)$	$-177 \div (-3)$	$-108 \div 18$
6.	$-132 \div (-4)$	$-115 \div 23$	$48 \div 16$
7.	$65 \div (-5)$	$-70 \div 35$	$-148 \div (-37)$
8.	$-119 \div 17$	$75 \div 15$	$63 \div (-7)$
9.	$72 \div (-3)$	$-96 \div (-24)$	$-135 \div 27$
10.	$-178 \div (-89)$	$108 \div (-12)$	$-121 \div (-11)$

1. What is the additive inverse of -8? of 8? of -13? of 15? 46

2. What replacement for w will make this sentence true? 47

$$w + (-7 + 7) = 6$$

3. Draw a number line and use arrows to find the sums. 50–52
a. $6 + (-3)$ **b.** $5 + (-7)$ **c.** $-3 + (-5)$ **d.** $1 + (-5)$

4. Find the sum of -11 and 3 by first expressing -11 as the sum of two negative integers, and then using the associative principle for addition. 52

5. Find the sum for each of the following examples. 50–52
a. $17 + 6$ **b.** $-37 + 25$ **c.** $16 + (-18)$ **d.** $-18 + (-17)$
e. $-5 + 6$ **f.** $-23 + 8$ **g.** $-61 + (-98)$ **h.** $67 + (-95)$

6. Order each of the following pairs, using the symbol "$<$." 53–54
a. $-7, 0$ **b.** $1, -5$ **c.** $-3, -4$ **d.** $-2, -11$

7. In each of the following, find the replacement for the question mark that makes the sentence a true statement. 55–56
a. $|6| = ?$ **b.** $|-6| = ?$ **c.** $|-13| = ?$ **d.** $|18| = ?$

8. If a is a positive integer and b is a negative integer, and $|b| > |a|$, then is the sum $a + b$ positive, or is it negative? 57–58

9. If x and y represent integers, finding the difference $x - y$ is the same as adding $\underline{?}$ to x. Subtract. 59–60
a. $4 - (-1)$ **b.** $-2 - 9$ **c.** $-3 - 3$ **d.** $-5 - (-13)$

10. Find the products. 62–65
a. $-4 \cdot 23$ **b.** $-5 \cdot (-16)$ **c.** $-31 \cdot 36$ **d.** $-7 \cdot (-19)$
e. $7 \cdot (-23)$ **f.** $-12 \cdot 11$ **g.** $-18 \cdot (-17)$ **h.** $21 \cdot (-11)$

11. Find the quotients. 66–67
a. $-18 \div 3$ **b.** $25 \div (-5)$ **c.** $-48 \div (-24)$ **d.** $24 \div (-8)$
e. $-39 \div 3$ **f.** $39 \div (-13)$ **g.** $-45 \div (-3)$ **h.** $52 \div (-13)$

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 397
Challenge Page 412

THE RATIONAL NUMBERS

We have seen that the integers are not closed under division. Let us therefore construct a set of numbers in which division (except division by zero) is always possible. We will denote the quotient of $1 \div 2$ as $\frac{1}{2}$; the quotient of $7 \div 5$ as $\frac{7}{5}$; the quotient of $-1 \div 2$ as $\frac{-1}{2}$; the quotient of $3 \div (-4)$ as $\frac{3}{-4}$; and so on.

From your experience in arithmetic, you recognize that $1 \div 2 = \frac{1}{2}$ and $7 \div 5 = \frac{7}{5}$ are true statements. You can also speak about the numbers $\frac{1}{2}$ and $\frac{7}{5}$ and know what you mean. But numbers such as $\frac{-1}{2}$ and $\frac{3}{-4}$ need more interpretation.

To begin with, we will give our set of numbers a name. Let us call them **rational numbers** and give the following definition.

▶ A rational number is a number that can be expressed as the quotient of two integers; that is, a number that can be expressed in the form $\frac{a}{b}$ (where a and b represent integers and $b \neq 0$).

When the rational number is expressed in "fraction" form, $\frac{a}{b}$, we speak (as in arithmetic) of a as the numerator of the fraction, b as its denominator, and a and b as its terms.

Why have we written $b \neq 0$?

1. Is it true that $9 = 9 \div 1$, or $\frac{9}{1}$? Is it true that $-3 = -3 \div 1$, or $\frac{-3}{1}$? If n represents any integer, is it true that $n = n \div 1$, or $\frac{n}{1}$? Then the set of integers may be regarded as a subset of the set of rational numbers.

2. What integer is named by $-8 \div 2$? $8 \div (-2)$? $-(8 \div 2)$? Then $\frac{-8}{2}$, $\frac{8}{-2}$, and $-\frac{8}{2}$ are all names for the same number, -4.

3. Are all of the following expressions names for the same number?

$$\frac{-15}{3} \qquad \frac{15}{-3} \qquad -\frac{15}{3}$$

What integer is named by these expressions?

4. Study your work in **2** and **3**. On the basis of this work, do you agree that the quotients $-1 \div 2$ and $1 \div (-2)$ —that is, $\frac{-1}{2}$ and $\frac{1}{-2}$ —name the same number, and that this number can be represented as $-\frac{1}{2}$? Express $\frac{-5}{4}$ in this form.

▶ If a and b represent positive integers:

$$-\frac{a}{b} = \frac{-a}{b} \text{ or } \frac{a}{-b}$$

We call a number such as $-\frac{1}{2}$ or $-\frac{5}{4}$ (expressed in any equivalent form) a **negative rational number.** A number such as -4 is also called a negative rational number.

5. Find the quotient for each of the following pairs of examples.

a. $12 \div 4$	$-12 \div (-4)$
b. $16 \div 8$	$-16 \div (-8)$
c. $27 \div 3$	$-27 \div (-3)$

In part **a**, do you obtain the same number as the quotient in each example? in part **b**? in part **c**?

6. On the basis of your work in **5**, do you agree that $\frac{11}{5}$ and $\frac{-11}{-5}$ ought to name the same number? $\frac{-2}{-3}$ and $\frac{2}{3}$?

▶ In symbols, we say: If a and b are positive integers:

$$\frac{a}{b} = \frac{-a}{-b}$$

A number such as $\frac{1}{2}$ or $\frac{5}{4}$ is called a **positive rational number.** A number such as 17 is also called a positive rational number.

We may summarize as follows:

▶ The rational number $\frac{a}{b}$ is positive if both a and b are positive or if both are negative. The rational number $\frac{a}{b}$ is negative if only one of its terms is negative and the other is positive.

EXERCISES

In Exercises **1–4**, tell which of the numbers named are positive rational numbers and which are negative rational numbers.

	a	**b**	**c**	**d**	**e**
1.	$\frac{3}{-7}$	$\frac{-5}{9}$	$\frac{-9}{-4}$	$\frac{-7}{8}$	$\frac{-6}{-11}$
2.	$\frac{15}{-13}$	$\frac{-1}{-6}$	$\frac{-2}{5}$	$-\frac{4}{3}$	$\frac{-3}{-8}$

	a	b	c	d	e
3.	$\frac{-5}{-16}$	$\frac{7}{-23}$	$\frac{-19}{-32}$	$\frac{-9}{10}$	$\frac{-10}{9}$
4.	$\frac{13}{-14}$	$\frac{-8}{-5}$	$-\frac{2}{7}$	$\frac{-17}{-48}$	$\frac{3}{-64}$

Multiplication of Rational Numbers

In defining operations in the set of rational numbers, we will proceed as we did when we defined an operation such as multiplication in the set of integers. That is, we will maintain the principles that governed operations in the set of integers. Also, we will define our operations so that the behavior of "fractions"—as we remember it from arithmetic—is preserved in the set of rational numbers.

1. Let us begin with the operation of multiplication. To find the product of $\frac{2}{3}$ and $\frac{4}{7}$, you know you proceed as follows: $\frac{2}{3} \cdot \frac{4}{7} = \frac{8}{21}$.

▶ In symbols: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad (b \neq 0,\ d \neq 0)$

2. Now, in the set of rational numbers, let us agree that multiplication is defined as shown above. Then:

A. $\frac{-4}{2} \cdot \frac{15}{5} = \frac{-60}{10}$ **B.** $\frac{-3}{-4} \cdot \frac{7}{-8} = \frac{-21}{32}$

In illustration **A**, we can check to see if we have been guilty of any inconsistency. What integer is named by $\frac{-4}{2}$? What integer is named by $\frac{15}{5}$? What integer is named by $\frac{-60}{10}$? In the set of integers, does $-2 \cdot 3$ equal -6?

Of course, in **A**, we chose rational numbers that could be re-expressed as integers. So this is a specialized example. Most examples you encounter will not be of this kind. However, such an example does indicate that the definition is reasonable.

EXERCISES

In each of the following, multiply, using the agreement $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ to find the product. Then, in whichever examples it is possible, check by interpreting the multiplier, multiplicand, and product as integers.

1. $\frac{1}{-2} \cdot \frac{-5}{12}$ 2. $\frac{5}{-6} \cdot \frac{-7}{-8}$ 3. $\frac{-9}{-10} \cdot \frac{-3}{-7}$

4. $\frac{3}{2} \cdot \frac{-5}{-8}$ 5. $\frac{-18}{-6} \cdot \frac{8}{2}$ 6. $\frac{-2}{-3} \cdot \frac{-7}{15}$

7. $\frac{12}{4} \cdot \frac{25}{-5}$ 8. $\frac{-6}{-7} \cdot \frac{8}{7}$ 9. $\frac{-9}{3} \cdot \frac{-21}{7}$

10. $\frac{-8}{11} \cdot \frac{9}{-5}$ 11. $\frac{-3}{-4} \cdot \frac{-5}{2}$ 12. $\frac{-10}{-2} \cdot \frac{35}{-5}$

13. $\frac{7}{-4} \cdot \frac{3}{11}$ 14. $\frac{10}{-13} \cdot \frac{-8}{7}$ 15. $\frac{8}{-4} \cdot \frac{20}{-5}$

16. $\frac{-11}{-12} \cdot \frac{1}{-3}$ 17. $\frac{6}{-3} \cdot \frac{-20}{4}$ 18. $\frac{-1}{-6} \cdot \frac{-7}{-4}$

19. $\frac{-12}{3} \cdot \frac{16}{-4}$ 20. $\frac{-8}{5} \cdot \frac{-3}{-7}$ 21. $\frac{15}{-3} \cdot \frac{12}{-2}$

22. $\frac{5}{9} \cdot \frac{5}{-6}$ 23. $\frac{-3}{13} \cdot \frac{-3}{8}$ 24. $\frac{3}{7} \cdot \frac{-2}{5}$

Identity Element for Multiplication

As we did when we defined the operation of multiplication for integers, we will maintain the number 1 as the identity element for multiplication. The number 1 can be easily expressed in the form $\frac{a}{b}$ as $\frac{1}{1}$. Why?

1. What is the quotient for each of the following: $-1 \div (-1)$? $4 \div 4$? $-3 \div (-3)$? $-11 \div (-11)$?

Then do $\frac{-1}{-1}$, $\frac{4}{4}$, $\frac{-3}{-3}$, and so on, all name the identity element in the set of rational numbers? We say:

▶ For all replacements of x by integers (other than zero):

$\frac{x}{x}$ names the identity element 1. Or: $\frac{x}{x} = 1$.

2. In arithmetic, when you were asked to express a fraction in *higher terms*—for example, to express $\frac{7}{8}$ as twenty-fourths—you may have thought "$24 \div 8 = 3$. Therefore, I will multiply both numerator and denominator by 3."

Do you see (illustration **A**) that you were really using $\frac{3}{3}$ as the identity element for multiplication and finding a new name for $\frac{7}{8}$?

In expressing a fraction such as $\frac{65}{80}$ in *lowest terms*, your work might have appeared as shown in illustration **B**. How is the identity element used in this work?

A.
$$\frac{7}{8} = \frac{?}{24}$$
$$\frac{7}{8} = \frac{3 \cdot 7}{3 \cdot 8}$$
$$= \frac{21}{24}$$

B.
$$\frac{65}{80} = \frac{5 \cdot 13}{5 \cdot 16}$$
$$= \frac{13}{16}$$

Common Name: Rational Numbers

Using the identity element, we can express a rational number in an unlimited number of ways. However, we may speak of the *common name* of a rational number $\frac{a}{b}$, using the familiar idea of lowest terms. That is, a and b should have no common factors other than 1.

The common name of $\frac{-4}{6}$ is "negative $\frac{2}{3}$" and is written either as $\frac{-2}{3}$ or as $-\frac{2}{3}$. The common name of $\frac{-4}{-5}$ is "positive $\frac{4}{5}$" since $\frac{-4}{-5} = \frac{-1 \cdot 4}{-1 \cdot 5}$ and $\frac{-1}{-1}$ is simply the identity element.

The common name of a number such as $\frac{5}{1}$ or $\frac{-3}{1}$ is simply 5 or -3, and you can easily use it in this form when you multiply. That is, you write $5 \cdot \frac{1}{7} = \frac{5}{7}$ (just as you always have) or $-3 \cdot \frac{1}{5} = -\frac{3}{5}$.

Inverse Elements for Multiplication

You know that we speak of a pair of elements as inverse elements for a given operation if, by applying the operation to this pair of elements, we obtain the identity element for the operation as the result.

1. Suppose we ask: "Does $\frac{1}{2}$ have an inverse element for multiplication—a **multiplicative inverse**—in the set of rational numbers?" We are really asking: In the set of rational numbers, is there a replacement for the variable that will make the following sentence true?

$$\tfrac{1}{2} \cdot x = 1$$

What number is the multiplicative inverse of $\frac{1}{2}$?

2. Find the solution set of each of the following sentences, assuming that the replacement set is the set of rational numbers.

$$\tfrac{1}{8} \cdot y = 1 \qquad \tfrac{3}{4} \cdot z = 1 \qquad \tfrac{7}{5} \cdot q = 1 \qquad 3 \cdot x = 1$$

What is the multiplicative inverse of $\frac{1}{8}$? of $\frac{3}{4}$? of $\frac{7}{5}$? of 3?

By now you have probably recognized that you know another name for the multiplicative inverse of a number.

▶ If the product of a pair of numbers is 1, the numbers are called **reciprocals** of each other.

3. Find the solution set of each of the following sentences, if the replacement set is the set of rational numbers. (Remember that the product must be 1, *not* -1.)

$$\tfrac{-1}{7} \cdot m = 1 \qquad \tfrac{3}{-5} \cdot n = 1 \qquad -1 \cdot r = 1 \qquad -2 \cdot r = 1 \qquad \tfrac{-11}{8} \cdot x = 1$$

What is the multiplicative inverse, or reciprocal, of $-\frac{1}{7}$? $-\frac{3}{5}$? -1? -2? $-\frac{11}{8}$?

4. Look for a pattern in your work in **2** and **3**. What kind of number—positive or negative—is the reciprocal of a positive rational number? What kind of number—positive or negative—is the reciprocal of a negative rational number?

Division

For each of the following pairs of examples, first find the indicated quotient. Then find the indicated product.

1.	$48 \div 12$	$48 \cdot \frac{1}{12}$
2.	$24 \div (-2)$	$24 \cdot (-\frac{1}{2})$
3.	$-12 \div 6$	$-12 \cdot \frac{1}{6}$
4.	$-18 \div (-3)$	$-18 \cdot (-\frac{1}{3})$

Did you find that in each pair of examples the quotient and the product were equal? Do these pairs of examples seem consistent with a principle that you used in earlier years; namely, the quotient of two numbers is equal to the product of the dividend and the reciprocal of the divisor?

We have not *proved* that the principle is true. We have simply examined some special division examples (dividend and divisor both integers) and found that the quotient can be obtained by multiplication in the set of rational numbers. Let us, however, agree to accept the principle as true for all rational numbers. (Division by zero not permitted.) In mathematical symbolism:

▶ For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0,\ c \neq 0,\ d \neq 0)$:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

In other words, dividing by a number is the same as multiplying, using its reciprocal, or multiplicative inverse.

5. a. To find the quotient $-26 \div \frac{2}{5}$, you find the product $-26 \cdot \frac{5}{2}$. What answer do you obtain?

b. To find the quotient $\frac{1}{3} \div \frac{-2}{9}$, you find the product $\frac{1}{3} \cdot \frac{9}{-2}$. What answer do you obtain?

EXERCISES

Find the quotients by multiplying, using the reciprocals of the divisors. Express each quotient by the common name.

1. $\frac{1}{3} \div \frac{-2}{3}$
2. $-16 \div \frac{4}{5}$
3. $\frac{-3}{4} \div \frac{7}{-8}$
4. $\frac{-7}{8} \div 7$
5. $\frac{-2}{-3} \div \frac{-3}{4}$
6. $32 \div \frac{-16}{25}$
7. $\frac{15}{16} \div (-5)$
8. $\frac{7}{8} \div \frac{2}{3}$
9. $\frac{-5}{-7} \div \frac{-1}{-3}$
10. $-12 \div \frac{-6}{7}$
11. $\frac{-9}{10} \div \frac{3}{5}$
12. $\frac{3}{4} \div 8$
13. $\frac{-2}{5} \div \frac{-5}{6}$
14. $-21 \div \frac{7}{8}$
15. $\frac{3}{7} \div \frac{-9}{-14}$
16. $\frac{-5}{12} \div 3$
17. $9 \div \frac{2}{-3}$
18. $\frac{4}{-9} \div \frac{-5}{-8}$
19. $\frac{-12}{-13} \div \frac{-4}{-7}$
20. $\frac{33}{13} \div (-11)$
21. $\frac{3}{4} \div \frac{-2}{11}$
22. $\frac{2}{-3} \div (-3)$
23. $\frac{5}{-6} \div \frac{-5}{21}$
24. $-5 \div \frac{-3}{5}$

Commutative Group under Multiplication

1. Examine the sentence $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ $(b \neq 0, d \neq 0)$, which is our definition of multiplication in the set of rational numbers. Are a and c integers? Then the product ac must also be an integer. Why? Are b and d non-zero integers? Then bd must also be a non-zero integer.

Then $\frac{ac}{bd}$ is the quotient of two integers. By definition, then, $\frac{ac}{bd}$ must be a rational number.

Does this reasoning indicate that the product of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is a rational number? Then the set of rational numbers is closed under multiplication.

2. For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$, would it be true that $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$? Use reasoning similar to that used in 1 to analyze the question. (In this case, remember that multiplication of integers is commutative.) Is multiplication of rational numbers commutative? Work out three examples, using pairs chosen from these numbers: $\frac{-1}{2}, \frac{2}{3}, \frac{-6}{7}, \frac{9}{5}, \frac{-15}{16}, \frac{9}{-2}, \frac{16}{3}$.

3. Do you think that multiplication of rational numbers is associative? That is, do you think it is true that:

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) \qquad (b \neq 0,\ d \neq 0,\ f \neq 0)$$

Work out three examples, choosing from the numbers named in **2.**

4. Name the rational number that is represented by $\frac{0}{d}$ if d is replaced by 1; if d is replaced by -3; by -6. For any non-zero replacement of d, what does $\frac{0}{d}$ equal?

Does 0 have a multiplicative inverse? For example, can you find a rational number $\frac{a}{b}$ such that:

$$\frac{0}{d} \cdot \frac{a}{b} = 1 \qquad (d \neq 0,\ b \neq 0)$$

You know that $0 \cdot a$ equals 0, and $d \cdot b$ equals db. Then we have:

$$\frac{0}{db} = 1$$

But $\frac{0}{db}$ cannot equal 1. Why? Therefore, 0 has no reciprocal.

5. If we exclude zero, does every rational number have a multiplicative inverse? Is division, except by zero, always possible in the set of rational numbers?

Study your results in **1–5** and examine the list of properties of a commutative group, as given on page 62. Do you agree that the rational numbers (zero excluded) form a commutative group under the operation of multiplication?

6. Try this one by yourself. Property by property, test to see if the set of integers, zero excluded $\{1, -1, 2, -2, 3, -3, \ldots\}$, is a commutative group under the operation of multiplication. Remember, for a set to be a commutative group under multiplication, the set must be closed under multiplication; multiplication must be commutative and associative; there must be an identity element for multiplication within the set; and each element must have an inverse element for multiplication within the set.

Which property, if any, is not satisfied? Is division always possible in the set $\{1, -1, 2, -2, 3, -3, \ldots\}$?

Does it seem that when not every member of a set has an *inverse element for a given operation,* then the *inverse operation* is not always possible in the set? Refer back to the set $\{0, 1, 2, 3, \ldots\}$ and the operation of *addition* for help in your answer.

Addition of Rational Numbers

In the addition of fractions in arithmetic, when two fractions had the same denominator, you performed the addition quite easily; for example, $\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$. Describe the procedure.

1. The illustrations below show a similar procedure used in the set of rational numbers.

A. $\frac{-6}{2} + \frac{-8}{2} = \frac{-6 + (-8)}{2}$
$= \frac{-14}{2}$
$= -7$

B. $\frac{-15}{3} + \frac{12}{3} = \frac{-15 + 12}{3}$
$= \frac{-3}{3}$
$= -1$

In each example, express the addends as integers and perform the addition in the set of integers. Are the results consistent?

Let us agree to accept the following principle as true in the set of rational numbers.

▶ For any two rational numbers $\frac{a}{c}$ and $\frac{b}{c}$ $(c \neq 0)$, $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$.

Then $\frac{-2}{7} + \frac{5}{7} = \frac{-2 + 5}{7}$, or $\frac{3}{7}$.

2. Find the sums.

a. $\frac{-7}{8} + \frac{5}{8}$

b. $\frac{12}{13} + \frac{-7}{13}$

c. $\frac{-3}{16} + \frac{-5}{16}$

d. $\frac{-28}{4} + \frac{8}{4}$

e. $\frac{2}{11} + \frac{-8}{11}$

f. $\frac{-7}{25} + \frac{-6}{25}$

g. $\frac{45}{5} + \frac{-30}{5}$

h. $\frac{-4}{9} + \frac{8}{9}$

3. What is the sum of $\frac{2}{9}$ and $\frac{-2}{9}$? the sum of $\frac{5}{7}$ and $\frac{-5}{7}$? the sum of $\frac{-3}{4}$ and $\frac{3}{4}$? What name can you give to each of these pairs? Do you think that every rational number $\frac{a}{b}$ $(b \neq 0)$ has an additive inverse? How might we represent the additive inverse of $\frac{a}{b}$? What number is the additive inverse of 0?

4. In arithmetic, when you were asked to find the sum of two fractions with different denominators—for instance, $\frac{2}{5}$ and $\frac{1}{7}$—you found a way to express the fractions as fractions with a common denominator. Perhaps you were not aware of it, but again you made use of the identity element of multiplication. After deciding that 35 was a common denominator, you expressed $\frac{2}{5}$ as $\frac{14}{35}$, and $\frac{1}{7}$ as $\frac{5}{35}$, and then added. What names did you use for the identity element?

$\frac{2}{5} + \frac{1}{7} = \frac{2}{5} \cdot \frac{7}{7} + \frac{1}{7} \cdot \frac{5}{5}$
$= \frac{14}{35} + \frac{5}{35}$, or $\frac{19}{35}$

5. Similarly, to add $\frac{-6}{3}$ and $\frac{10}{2}$, we can make use of the identity element. We need only recognize that 6 is a common denominator.

What integer is named by $\frac{-6}{3}$? by $\frac{10}{2}$? In the set of integers, is it true that $-2 + 5 = 3$?

$\frac{-6}{3} + \frac{10}{2} = \frac{-6}{3} \cdot \frac{2}{2} + \frac{10}{2} \cdot \frac{3}{3}$
$= \frac{-12}{6} + \frac{30}{6}$
$= \frac{-12 + 30}{6}$
$= \frac{18}{6}$, or 3

6. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$ are any two rational numbers whose sum we are to determine. Will bd represent a common denominator? Will $\frac{a}{b} \cdot \frac{d}{d}$ and $\frac{c}{d} \cdot \frac{b}{b}$ yield the numbers re-expressed with a common denominator? Then, using our definition of addition (and applying the commutative principle), we can proceed as shown at the right.

$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b}$
$= \frac{ad}{bd} + \frac{cb}{db}$
$= \frac{ad + cb}{bd}$

7. What is the sum of $\frac{-3}{5}$ and 0? You can probably give your answer quite easily: "$\frac{-3}{5}$." Let us, however, see if this can be explained using our definition concerning addition in the rational numbers.

We can begin by writing 0 as $\frac{0}{5}$. Why is this possible? Then $\frac{-3}{5} + \frac{0}{5} = \frac{-3 + 0}{5}$, or $\frac{-3}{5}$. Use a similar method and find the sum $\frac{-5}{2} + 0$; the sum $0 + \frac{7}{3}$. Do you agree that 0 is the identity element for addition in the set of rational numbers?

Subtraction

On pages 59–60, you found that the operation of subtraction could be defined in terms of addition. That is, you found that the difference $x - y$ could be defined as the sum $x + (-y)$. This relationship was developed for integers; however, since every rational number has an additive inverse, or opposite, perhaps it is possible to define subtraction as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} \qquad (b \neq 0,\ d \neq 0)$$

1. Consider illustration **A** below. Notice that $\frac{60}{5} - \frac{-35}{5}$ has been expressed as the sum of $\frac{60}{5}$ and the additive inverse of $\frac{-35}{5}$; then the sum has been found, using the addition principles. A similar procedure has been followed in **B**.

A.
$$\frac{60}{5} - \frac{-35}{5} = \frac{60}{5} + \frac{35}{5}$$
$$= \frac{60 + 35}{5}$$
$$= \frac{95}{5}, \text{ or } 19$$

B.
$$\frac{-36}{6} - \frac{12}{6} = \frac{-36}{6} + \frac{-12}{6}$$
$$= \frac{-36 + (-12)}{6}$$
$$= \frac{-48}{6}, \text{ or } -8$$

In each case, check the results for consistency, using addition in the set of integers.

2. Consider a simple example from arithmetic: $\frac{7}{3} - \frac{2}{3} = \underline{?}$. You can give the answer immediately: "$\frac{5}{3}$." Do you obtain the same result by finding the sum $\frac{7}{3} + \frac{-2}{3}$? Then here, too, the relationship seems consistent with earlier experiences.

Let us therefore accept as true the following principle.

▶ For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0,\ d \neq 0)$:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

You will notice that in our illustrations we have used a special case—namely, the case where $\frac{a}{b}$ and $\frac{c}{d}$ have the same denominator; that is, $b = d$. But you already know how to re-express rational numbers so that they have a common denominator; therefore, an example such as $\frac{1}{3} - \frac{-2}{5}$ should cause you no difficulty.

EXERCISES

Perform the indicated operations.

1. $\frac{-5}{7} + \frac{2}{7}$	**2.** $\frac{-5}{12} - \frac{-7}{12}$	**3.** $\frac{4}{5} + \frac{-1}{6}$
4. $\frac{-3}{5} - \frac{5}{8}$	**5.** $\frac{7}{10} + \frac{-3}{10}$	**6.** $\frac{-5}{8} - \frac{7}{8}$
7. $\frac{-5}{8} + \frac{-3}{20}$	**8.** $\frac{1}{4} - \frac{13}{17}$	**9.** $\frac{-11}{15} + \frac{-2}{15}$
10. $\frac{7}{16} - \frac{-3}{16}$	**11.** $\frac{-1}{4} + \frac{7}{8}$	**12.** $\frac{-15}{19} - \frac{-1}{2}$
13. $\frac{-1}{6} + \frac{5}{6}$	**14.** $\frac{3}{11} - \frac{9}{11}$	**15.** $\frac{-9}{10} + \frac{-11}{15}$
16. $\frac{-3}{7} - \frac{5}{9}$	**17.** $\frac{5}{12} + \frac{-11}{12}$	**18.** $\frac{-4}{9} - \frac{7}{9}$
19. $\frac{-3}{4} + \frac{2}{3}$	**20.** $\frac{2}{3} - \frac{-7}{13}$	**21.** $\frac{-3}{17} + \frac{-4}{17}$
22. $\frac{-11}{14} - \frac{-9}{14}$	**23.** $\frac{2}{9} + \frac{-7}{12}$	**24.** $\frac{-5}{6} - \frac{-10}{11}$

Commutative Group under Addition

1. Is the sum $\frac{-1}{2} + \frac{-2}{3}$ a rational number? Is the sum $\frac{-7}{4} + \frac{3}{8}$ a rational number? the sum $\frac{-9}{5} + \frac{6}{5}$? Do you think that the rational numbers are closed under the operation of addition?

2. Is the sum $\frac{-2}{3} + \frac{-1}{2}$ equal to the sum $\frac{-1}{2} + \frac{-2}{3}$ (found in 1)? Is the sum $\frac{3}{8} + \frac{-7}{4}$ equal to the sum $\frac{-7}{4} + \frac{3}{8}$? Is the sum $\frac{6}{5} + \frac{-9}{5}$ equal to the sum $\frac{-9}{5} + \frac{6}{5}$? Do you think that addition of rational numbers is commutative?

3. Find the sum in each of the following pairs of examples.

a. $\left(\frac{-1}{4} + \frac{1}{2}\right) + \frac{3}{8}$ $\qquad$ $\frac{-1}{4} + \left(\frac{1}{2} + \frac{3}{8}\right)$

b. $\left(\frac{2}{3} + \frac{-1}{6}\right) + \frac{-7}{6}$ $\qquad$ $\frac{2}{3} + \left(\frac{-1}{6} + \frac{-7}{6}\right)$

c. $\left(\frac{-3}{5} + \frac{1}{10}\right) + \frac{-2}{5}$ $\qquad$ $\frac{-3}{5} + \left(\frac{1}{10} + \frac{-2}{5}\right)$

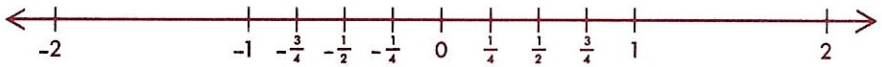
Do you think that addition of rational numbers is associative?

You already know that 0 is the identity element for the operation of addition in the set of rational numbers. You know also that every rational number $\frac{a}{b}$ $(b \neq 0)$ has an additive inverse that can be represented as $\frac{-a}{b}$.

Then are the rational numbers a commutative group under the operation of addition? Is subtraction always possible in the set of rational numbers? Give a reason for your answer. (Use the agreement concerning subtraction that was made on page 80.)

Order of Rational Numbers

On page 49, we formed a number line and marked points on it corresponding to the integers. Marked on the number line below are points corresponding to 0, the positive rational numbers $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, and 2, and the negative rational numbers $-\frac{1}{4}$, $-\frac{1}{2}$, $-\frac{3}{4}$, -1, and -2.



Notice that the points corresponding to the positive rational numbers are to the right of 0, and the points corresponding to the negative rational numbers are to the left of 0. Does it seem reasonable that the points corresponding to $\frac{1}{2}$ and $-\frac{1}{2}$ are at an equal distance from 0?

1. Suppose you were asked to mark a point corresponding to $\frac{3}{2}$ and a point corresponding to $-\frac{3}{2}$. In marking such points, you may find it convenient to think of other names for $\frac{3}{2}$ and $-\frac{3}{2}$. You know that $\frac{3}{2}$ can be expressed as $1\frac{1}{2}$ or 1.5; similarly, $-\frac{3}{2}$ can be expressed as $-1\frac{1}{2}$ or -1.5. (You will learn more about rational numbers and their decimal representation in Chapter 8.) Where should the point corresponding to $\frac{3}{2}$ be marked? the point corresponding to $-\frac{3}{2}$?

2. Draw a number line and mark the points corresponding to the following set of numbers: $\{-2, -\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, \ldots, 2\}$. Express $-\frac{9}{5}$ in the two ways similar to those given for $-\frac{3}{2}$.

3. You have already learned about the order of integers and the number line. That is, for any integers a and b, you know that if a point corresponding to a is to the left of a point corresponding to b, then a is less than b $(a < b)$. Thus, $-2 < -1$, $-1 < 1$, and so on.

A similar statement is true in the set of rational numbers. The point corresponding to $-\frac{3}{2}$ is to the left of the point corresponding to $-\frac{1}{2}$, and we write: $-\frac{3}{2} < -\frac{1}{2}$. Is it true, then, that $-\frac{9}{5} < \frac{1}{5}$?

4. Using the number line, tell which of the following sentences are true statements.

a. $-\frac{7}{2} < -\frac{5}{2}$ **b.** $-\frac{1}{4} < -\frac{3}{4}$ **c.** $-\frac{7}{5} < -\frac{6}{5}$

Do you remember how "is less than" was *defined* on page 54? The definition holds for the set of rational numbers. A number x is less than a number y if a positive number must be added to x to obtain y.

It may seem strange that we have not used $\frac{a}{b}$ and $\frac{c}{d}$ to represent rational numbers, but in this case x and y are just as serviceable. We are not, after all, concerned with numerators or denominators.

5. In each of the following sentences, replace the variable by the rational number that makes the sentence a true statement.

a. $-\frac{7}{2} + q = -\frac{5}{2}$ **b.** $-\frac{1}{4} + r = -\frac{3}{4}$ **c.** $-\frac{7}{5} + s = -\frac{6}{5}$

Do the results confirm the conclusions you reached using the number line (**4a–c**)? Why is $-\frac{1}{4}$ *not less than* $-\frac{3}{4}$?

6. Suppose you are asked to order three numbers such as $-\frac{3}{4}$, 1, and $-2\frac{1}{2}$. Is it true that $-2\frac{1}{2} < -\frac{3}{4}$? $-\frac{3}{4} < 1$? Then we can write: $-2\frac{1}{2} < -\frac{3}{4}$ *and* $-\frac{3}{4} < 1$. We may also express this relationship as follows:

$$-2\tfrac{1}{2} < -\tfrac{3}{4} < 1$$

In each of the following, order the numbers, using the form $x < y < z$.

a. $\frac{3}{4}$, $-\frac{1}{2}$, $\frac{1}{2}$ **b.** -1, $-\frac{3}{5}$, $-\frac{4}{5}$ **c.** 0, $-\frac{1}{5}$, 2

7. As you know, the symbol ">" is also used to indicate order. Is 3 greater than -1? Is -1 greater than $-1\frac{1}{4}$? Then, using the symbol ">," we can write: $3 > -1$ *and* $-1 > -1\frac{1}{4}$. Write a statement of this relationship in the form $x > y > z$.

Using the same form, order the numbers in each of the following.

a. $\frac{1}{10}$, $\frac{3}{10}$, $-\frac{1}{10}$ **b.** -1, $-\frac{7}{6}$, $\frac{5}{6}$ **c.** $-\frac{11}{6}$, $-\frac{13}{6}$, -2

Betweenness

Consider the sentence "$-4 < -3$ *and* $-3 < -2$" ($-4 < -3 < -2$). Thinking of the number line, we can say: "-4 is to the left of -3 *and* -3 is to the left of -2." Do you agree that we can also say: -3 is between -4 and -2?

Do you think we can say: 3 is between 2 and 4? -6 is between -7 and -5? $\frac{1}{2}$ is between 0 and 1?

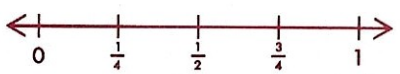
Let us find a precise interpretation of what we mean mathematically when we say that a number is *between* two other numbers.

You know that the relationship "-4 is less than -3" can be stated as "-3 is greater than -4." Then, if we use -3 as the subject of our original sentence ($-4 < -3 < -2$), we can say: "-3 is greater than -4 *and* less than -2."

Is 3 greater than 2 and less than 4? Is -6 greater than -7 and less than -5? Then let us make our definition as follows.

▶ If a number a is greater than a number b and less than a number c, then a is said to be between b and c.

Density



You know that $\frac{3}{4}$ is between $\frac{1}{2}$ and 1. Do you think there is a rational number between $\frac{3}{4}$ and 1?

Find the number named by the expression $(\frac{3}{4} + 1) \div 2$. Is this number greater than $\frac{3}{4}$ and less than 1? Then $\frac{7}{8}$ is a number between $\frac{3}{4}$ and 1.

Do you think there is a rational number between $\frac{7}{8}$ and 1? Find the number named by the expression $(\frac{7}{8} + 1) \div 2$. Is this number between $\frac{7}{8}$ and 1?

In a similar way, find a rational number between $1\frac{5}{16}$ and 1.

Do you see that when we add two numbers such as $3\frac{1}{32}$ and 1 and divide the sum by 2 we are taking the "average" of the numbers? The resulting number must be greater than one of the numbers and less than the other. Since the sum of two rational numbers is always a rational number, and since the rational numbers (zero excluded) are closed under division, will we always obtain a rational number as the result?

Do you think there is any limit to the number of times we can continue this procedure? Then is there a rational number between any two rational numbers you can name?

In mathematics, there is a special word that is used to describe the fact that there is always a rational number between any two rational numbers. We say that the set of rational numbers is **dense**, or has the property of **density**.

Suppose we consider only the set of integers. Is there an integer between any two integers you can name? For instance, is there an integer that is greater than 3 and less than 4? an integer that is greater than -6 and less than -5? Then we say that the set of integers *does not* have the property of density.

EXERCISES

1. Find a rational number that is between 4 and $4\frac{1}{7}$.

2. Find a rational number that is between $-\frac{1}{5}$ and $-\frac{1}{6}$. Is this number greater than $-\frac{1}{6}$, or is it less than $-\frac{1}{6}$? Is it greater than $-\frac{1}{5}$, or is it less than $-\frac{1}{5}$?

3. Find a rational number that is between $\frac{1}{99}$ and $\frac{1}{100}$.

4. Is there more than one rational number that is between 4 and $4\frac{1}{7}$ (Exercise **1**)? If you think there is only one (the one you have named), explain why you think there are no others. If you think there are other rational numbers between 4 and $4\frac{1}{7}$, then find at least three.

Do you think there is any limit to the number of rational numbers that are between two given rational numbers?

5. Find a number between 1.0008 and 1.0009.

6. As you know from your earlier courses in mathematics, an expression such as $\frac{5\frac{1}{6}}{18}$ may be called a **complex fraction.** You can now think of such an expression as the indicated quotient of two rational numbers. On page 77, you found that division (except by zero) is always possible in the set of rational numbers. Perform the indicated division: $5\frac{1}{6} \div 18$. Is the quotient a rational number?

7. Find what $\frac{-3\frac{1}{8}}{11}$ equals.

8. Find what $\frac{1}{4\frac{1}{4}}$ equals. What does $\frac{1}{4\frac{1}{4}} \cdot 4\frac{1}{4}$ equal?

9. Find what $\frac{1}{-3\frac{1}{2}}$ equals. What does $\frac{1}{-3\frac{1}{2}} \cdot \left(-3\frac{1}{2}\right)$ equal?

10. For any non-zero rational number x, do you think x and $\frac{1}{x}$ are reciprocals (multiplicative inverses)?

11. Are $\frac{(-3)^2}{(-7)^2}$ and $\frac{(-7)^2}{(-3)^2}$ reciprocals? What does $\frac{(-3)^2}{(-7)^2}$ equal?

12. What does $\left(\frac{-4}{5}\right)^2$ equal? $\left(\frac{-4}{5}\right)^3$? $\left(\frac{-4}{5}\right)^4$?

13. What does $\left(\frac{-1}{2}\right)^2$ equal? $\left(\frac{-1}{2}\right)^3$? $\left(\frac{-1}{2}\right)^4$? $\left(\frac{-1}{2}\right)^5$?

14. If a is any negative rational number, is a^2 positive, or is it negative? a^3? a^4? a^5?

15. Complete the following statements with *positive* or *negative*:
Any negative number raised to an even power is a ? number.
Any negative number raised to an odd power is a ? number.
Do we need two statements, or just one, for powers of positive numbers? Explain.

A Closing Note: The Distributive Principle

On page 72, as we prepared to define operations with rational numbers, one of the points made was that we wanted to have multiplication distributive over addition. So far, we have not had to test the distributive principle. Below are four sentences. Complete the indicated operations and tell which sentences are true.

1. $2 \cdot \left(\frac{1}{7} + \frac{-6}{7}\right) = \left(2 \cdot \frac{1}{7}\right) + \left(2 \cdot \frac{-6}{7}\right)$

2. $\frac{3}{4} \cdot \left[\frac{1}{2} + (-3)\right] = \left[\frac{3}{4} \cdot \frac{1}{2}\right] + \left[\frac{3}{4} \cdot (-3)\right]$

3. $-2 \cdot \left[\frac{-1}{3} + \frac{-2}{3}\right] = \left[-2 \cdot \left(\frac{-1}{3}\right)\right] + \left[-2 \cdot \left(\frac{-2}{3}\right)\right]$

4. $\frac{-5}{2} \cdot \left[\frac{4}{3} + \frac{6}{5}\right] = \left(\frac{-5}{2} \cdot \frac{4}{3}\right) + \left(\frac{-5}{2} \cdot \frac{6}{5}\right)$

Do you agree that multiplication is distributive over addition in the set of rational numbers?

5. Do you think that multiplication is distributive over subtraction in the set of rational numbers? That is, for any rational numbers a, b, and c, is the following statement true, or is it false?

$$a \cdot (b - c) = ab - ac$$

You may try a few examples if you wish. But for more effective reasoning, turn back to page 80 and look at the definition of subtraction. If multiplication is distributive over addition, must it also be distributive over subtraction?

1. A rational number is a number that can be expressed as the ? of two integers. Is -7 a rational number? 70–71

2. Which of the following are positive rational numbers? Which are negative rational numbers? 70–71

a. $\frac{-5}{-7}$ b. $\frac{-8}{11}$ c. $\frac{13}{-9}$ d. $\frac{-6}{-25}$ e. $\frac{17}{16}$ f. $-\frac{4}{19}$

3. Multiply. 72

a. $\frac{4}{-5} \cdot \frac{-3}{11}$ b. $\frac{-7}{-8} \cdot \frac{-3}{-5}$ c. $\frac{12}{3} \cdot \frac{15}{-5}$ d. $\frac{-3}{-4} \cdot \frac{-5}{13}$

e. $\frac{-5}{6} \cdot \frac{5}{7}$ f. $\frac{27}{-9} \cdot \frac{-16}{-2}$ g. $\frac{-2}{-3} \cdot \frac{-7}{-17}$ h. $\frac{8}{9} \cdot \frac{-4}{19}$

4. What is the multiplicative inverse, or reciprocal, of: 74–75

a. $-\frac{2}{5}$ b. $\frac{1}{4}$ c. $\frac{8}{3}$ d. -7 e. $\frac{5}{6}$ f. $-\frac{9}{8}$

Divide. 75

	a	b	c	d
5.	$\frac{1}{2} \div \frac{-2}{3}$	$20 \div \frac{-5}{-6}$	$\frac{5}{-9} \div \frac{-4}{-7}$	$\frac{-2}{-7} \div \frac{4}{-3}$
6.	$\frac{8}{15} \div (-4)$	$-10 \div \frac{3}{5}$	$\frac{9}{-11} \div 3$	$\frac{-4}{-5} \div \frac{-4}{-15}$

Add. 78–79

7.	$\frac{-2}{5} + \frac{3}{5}$	$\frac{-1}{2} + \frac{2}{5}$	$\frac{3}{8} + \frac{-5}{8}$	$\frac{7}{9} + \frac{-4}{9}$
8.	$\frac{2}{3} + \frac{-3}{7}$	$\frac{-5}{7} + \frac{2}{7}$	$\frac{-3}{4} + \frac{-5}{6}$	$\frac{-3}{10} + \frac{7}{12}$

Subtract. 80

9.	$\frac{-2}{7} - \frac{4}{7}$	$\frac{-1}{4} - \frac{3}{10}$	$\frac{5}{9} - \frac{-2}{9}$	$\frac{-7}{16} - \frac{-11}{16}$
10.	$\frac{-2}{5} - \frac{-5}{6}$	$\frac{3}{13} - \frac{9}{13}$	$\frac{4}{11} - \frac{-1}{3}$	$\frac{-2}{3} - \frac{3}{4}$

11. Order each set of numbers, using the symbol "<." 82–84

a. $\frac{1}{4}, -\frac{3}{4}$ b. $-\frac{1}{2}, \frac{1}{2}$ c. $-\frac{2}{3}, -2$

d. $-2\frac{1}{3}, 3\frac{1}{2}$ e. $-\frac{3}{8}, -\frac{5}{8}$ f. $-2\frac{1}{5}, -3\frac{1}{4}, 4\frac{1}{3}$

12. Because there is a rational number between any two rational numbers, the set of rational numbers is said to be ?. 84–85

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 398

Challenge Page 413

3 *Equations and Inequalities*

About the year 500, Hindu mathematicians in India became interested in solving problems such as the following.

> When traveling to market, a cotton merchant had to pay a tax at two toll stations. At the first, he paid $\frac{1}{10}$ of the number of bales he had. At the second, he paid $\frac{1}{9}$ of the bales he had left. After passing the two toll stations, he still had 16 bales of cotton. How many bales did he have originally?

In solving the problem, a Hindu mathematician might have written (using his own symbolism):

> Let $\square$ represent the number of bales the merchant had originally.
>
> Then $\frac{9}{10}\cdot\square$ represents the number of bales he had left after passing the first toll station.
>
> Then $\frac{8}{9}\cdot(\frac{9}{10}\cdot\square)$ represents the number of bales the merchant had left after passing the second toll station.
>
> The Hindu mathematician could then write: $\frac{8}{9}\cdot\frac{9}{10}\cdot\square = 16$.
>
> Can you find the number of bales the merchant had originally?

You recognize that the sentence $\frac{8}{9}\cdot\frac{9}{10}\cdot\square = 16$ is what we call an open sentence. What is the variable in the sentence? Is {20} the solution set of the sentence?

Through the study of algebra, the Hindus invented new tools for solving mathematical problems. Over the centuries, one very important part of algebra has been writing open sentences and solving them to obtain desired information. In this chapter and in later chapters, you will work with systematic methods for finding the solution sets of open sentences. These methods make use of the properties of numbers and the definitions of operations that you studied in preceding chapters.

SOLVING EQUATIONS AND INEQUALITIES

Classification of Sentences

You are familiar with two types of open sentences. Sentences in the form $3x - 5 = 7$ and $x = x + 1$ are called **equations.** Such sentences are formed when two expressions are joined by an equal sign.

Open sentences that are formed by using a sign of inequality, such as $y + 8 < 15$ and $3 + y > 2$, are called **inequalities.** Sentences such as $3 + y \leq 2$ and $2m \geq -14$ are also inequalities.

As you know, the solution set of an open sentence contains all the numbers that make the sentence a true statement.

Using the replacement set $\{-2, -1, 0, 1, 2, 3, 4\}$, find the solution set of each of the following equations and inequalities. Remember, if no member of the replacement set will make the sentence a true statement, the solution set is the empty set, $\emptyset$.

	a	b
1.	$x + 38 = 40$	$y + 1 > 1$
2.	$m - 20 = -16$	$a - 3 < -2$
3.	$x + 1 = -3$	$15a = 45$
4.	$3m \geq 9$	$5b = -10$
5.	$13 + n = 14$	$k + 1 < -2$
6.	$x + 17 > 18$	$2y < -2$
7.	$b - 3 \leq -2$	$7n = 0$
8.	$a + 1 = 6$	$2y \geq 10$

9. In a sentence, the expression at the left and the expression at the right of the equal sign or the inequality symbol are called the **members** of the sentence.

What is the left member of the equation $3x + 5 = 11$? of the inequality $7x - 14 > 59$?

What is the right member of the inequality $5n + 15 \leq 4n + 16$? of the equation $6n + n = 8n + 10 - 2n$?

10. In a product such as $5x$, each of the factors is called the **coefficient** of the other factor. Thus, we say that 5 is the coefficient of x, and x is the coefficient of 5. What is the coefficient of n in each of the following expressions?

a. $19n$ **b.** $-72n$ **c.** n **d.** nm **e.** $n(20 - t)$

Replacement Sets and Solution Sets

Sometimes an open sentence contains two different variables. Consider, for example, the equation $x + y = 6$. As you know, in such cases each variable may have its own replacement set. Frequently, however, the replacement set for both variables is the same set of numbers.

1. Suppose that for the sentence $x + y = 6$ the replacement set for both variables is $\{0, 1, 2, 3, 4\}$. We want to know all possible replacements for x and y that will make the sentence a true statement. To determine these number pairs, begin by replacing x with 0. What must be the value of y to make the sentence true? Is 6 a member of the replacement set? Then do you see that when 0 is the replacement for x there is no allowable replacement for y that will result in a true statement?

2. Now use 1, 2, 3, and 4, in turn, as replacements for x. Find in each case whether a member of the replacement set used as a replacement for y makes the sentence true. To tabulate the number pairs of the solution set, let us agree that in each pair the first number is the replacement for x. Write each pair in parentheses, for example (2,4), and list all the pairs within braces.

3. Do you think that in an equation with two variables the same number may be used as the replacement for both variables? Is the number pair (3,3) a member of the solution set of the equation $x + y = 6$?

4. When you have tried 0, 1, 2, 3, and 4 as replacements for x in finding the solution set of this equation, must you also try each member of the replacement set as a replacement for y and find the corresponding replacement for x? Why not?

5. Using the same replacement set as in **1**, find the number pairs that make up the solution set of each of the following open sentences. List each pair in the order indicated beside each sentence.

a. $x + y = 2$ (x,y) **b.** $a - b = 3$ (a,b)
c. $r + s = 8$ (r,s) **d.** $0 = p - q$ (p,q)

6. Some sentences are true for all members of the replacement set; for example, is $a + 3 = 3 + a$ a true sentence for all members of the replacement set $\{-3, -1, 0, 1.5, 3, 4.1\}$?

Suppose the replacement set is $\{0, 1, 2, 3, \ldots\}$. Is the open sentence "$5 - x$ equals a positive whole number" true for all members of the replacement set? Since the replacement set is not limited, you cannot test all its members. However, if $5 - x$ is zero or a negative number when x is replaced by any one of the members of the replacement set, the sentence is not true for all members of the set. Is there at least one such replacement? Then is the sentence true for all members of the replacement set?

7. Which of the following sentences are true for all members of the replacement set $\{0, 1, 2, 3, \ldots\}$? Which are not true for any member of the replacement set?

a. $x + 1$ is a member of $\{0, 1, 2, 3, \ldots\}$. **b.** $6 - y = 6 + (-y)$
c. $5x$ is a positive whole number. **d.** $a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$
e. $y = y + 1$ **f.** $y = y \cdot 1$
g. $n \times 7 = 7 \times n$ **h.** $-4x + 4x = 0$
i. $(-2 + x) + 5 = -2 + (x + 5)$ **j.** $(6 + 3)q = 8q$

8. Examine the sentences above that are true for all members of the replacement set. Do they represent number principles with which you are already familiar? List the sentences that are true for all possible replacements and describe the principle that each illustrates.

Equivalent Open Sentences

Let the replacement set for the following exercises be the set of rational numbers.

1. What is the solution set of $x + 145 = 150$? What is the solution set of $x = 5$? Is the solution set of $x + 145 = 150$ also the solution set of $x = 5$? Is the solution set of $x = 5$ also the solution set of $x + 145 = 150$?

2. Is $\{8, -8\}$ the solution set of $x^2 = 64$? What is the solution set of $x^2 + 4 = 68$? Do both equations have the same solution set?

3. Find the solution set of each of the following.

a. $5x = -15$ **b.** $7x = -21$ **c.** $14x = -42$

What did you find was true of the three equations?

If two equations are such that the solution set of one is the same as the solution set of the other, the equations are called **equivalent equations.**

In each of the following, find the two equations in parts **a**, **b**, and **c** that are equivalent equations.

4.	**a.** $x + 7 = 14$	**b.** $x - 14 = 21$	**c.** $x - 7 = 0$
5.	**a.** $3a + 3 = 9$	**b.** $4a + 7 = 11$	**c.** $2a + 1 = 3$
6.	**a.** $b^2 = 9$	**b.** $b^2 + 9 = 25$	**c.** $b^2 + 2 = 11$
7.	**a.** $32s = 4$	**b.** $\frac{1}{2}s = \frac{1}{16}$	**c.** $24s = 8$
8.	**a.** $x^2 - 10 = 15$	**b.** $x + 15 = 20$	**c.** $x^2 + 5 = 30$
9.	**a.** $\lvert x \rvert = 3$	**b.** $x^2 = 9$	**c.** $x = -3$

The replacement set for **10–15** is $\{-3, -2, -1, 0, 1, 2, 3\}$.

10. What is the solution set of $x + 8 < 9$? What is the solution set of $x < 1$? Do both inequalities have the same solution set?

11. What is the solution set of $3x > 3$? of $3x - 3 > 0$? of $x > 1$? Are the solution sets the same?

If two inequalities are such that the solution set of one is the same as the solution set of the other, the inequalities are called **equivalent inequalities.**

In each of the following find the two inequalities that are equivalent.

12.	**a.** $t - 5 < -6$	**b.** $t + 5 < 6$	**c.** $t - 3 < -4$
13.	**a.** $8a > 2$	**b.** $a > -1$	**c.** $4a > -4$
14.	**a.** $b + 20 \leq 22$	**b.** $b + 12 \leq 14$	**c.** $b - 2 \leq 2$
15.	**a.** $2c + 2 \geq 4$	**b.** $4c \geq 12$	**c.** $2c \geq 2$

16. The importance of equivalent equations and inequalities is that you can solve complicated open sentences by finding simple equivalent open sentences in which the solution set is obvious.

For example, you would probably find the equation $5x - 3 = \frac{1}{2}(14 + 6x)$ difficult to solve. Can you, however, quickly obtain the solution set of $x = 5$? Suppose you know that the two equations are equivalent. What must be the solution set of the first?

In the following pages you will discover some systematic methods for obtaining equivalent open sentences—first for solving equations, then for solving inequalities. Always check your answers.

Solving Equations: The Addition Principle

In the following work let us agree that, unless a different replacement set is given, the replacement set for the variable in an open sentence is the set of rational numbers.

When you solve an equation, you find the solution set of the equation. The members of the solution set are called the **roots** of the equation. What is the root of $x + 5 = 9$? of $y - 3 = 9$? of $5z = 25$? What are the two roots of $a^2 = 49$? of $b^2 = 64$?

Suppose you wish to solve the equation $x - 4 = 11$. You know that if you can find an equivalent equation of the form $x = k$, where k is some number, the solution set will be obvious. An equivalent equation of the form $k = x$ will also, of course, make the solution set obvious. Let us try to discover a systematic method for finding such an equation.

1. What is the root of $x - 4 = 11$? What is the root of $x - 4 + 1 = 11 + 1$? of $x - 4 + (-3) = 11 + (-3)$? Are the second and third equations each equivalent to the first? Why?

Does it seem that adding the same number to each member of an equation results in an equivalent equation? Test your answer by finding the solution sets of the equations below. State whether the equations in parts **b** and **c** are equivalent to the equation in part **a**.

	a	b	c
2.	$a + 1 = 4$	$a + 1 + 3 = 4 + 3$	$a + 1 + (-1) = 4 + (-1)$
3.	$5 + j = 3$	$5 + j + 2 = 3 + 2$	$5 + j + (-5) = 3 + (-5)$
4.	$y - 6 = 1$	$y - 6 + 7 = 1 + 7$	$y - 6 + 6 = 1 + 6$

Examine the equation in **2c**. Can we write this equation as $a = 3$? Explain. Examine the equation in **3c**. Can we express this equation as $j = -2$? Express the equation in **4c** in a similar way.

In each case, you notice, we have obtained an equation of the form $x = k$, where k is some number. Is the root obvious in such an equation?

5. In **2c**, we operated on both members of $a + 1 = 4$ by adding -1 to each member. How is the -1 related to the 1 in the left member?

In **3c**, we operated on both members of $5 + j = 3$ by adding -5 to each member. How is the -5 related to the 5 in the member containing the variable?

Look at **4a**. Is the equation $y - 6 = 1$ simply another way of writing $y + (-6) = 1$? Then, in adding 6 to each member of $y - 6 = 1$, did we follow a procedure similar to the procedure used in **2** and **3**?

Do you see a pattern for finding equivalent equations of the form $x = k$ (or $k = x$); that is, the *simplest equivalent equation*?

▶ If the left and right members of an equation are operated on by adding the same number to each, an equivalent equation is obtained.

▶ If an equation of the form $x + b = c$ is operated on by adding the additive inverse of b to each member—that is, $x + b + (-b) = c + (-b)$—the equivalent equation will be of the form $x = k$, the simplest equivalent equation.

EXERCISES

Using the addition principle, find the root of each of the following.

	a	b	c
1.	$y + 46 = 52$	$m + \frac{1}{2} = 5$	$x - 437 = -529$
2.	$r - \frac{3}{4} = \frac{1}{2}$	$a - 33 = 341$	$s + 1\frac{1}{2} = 3\frac{1}{2}$
3.	$7.25 + a = 9.5$	$x + 6 = 1$	$m - 3.21 = 2.32$
4.	$48 = 36 + n$	$14 = a + 2$	$1.4 = t + 2.7$
5.	$p - 56.7 = -34.3$	$y - 0 = 17$	$b - \frac{7}{8} = \frac{13}{16}$
6.	$132 + x = -12$	$m + \frac{1}{3} = 4\frac{1}{3}$	$p - 7 = 15$
7.	$1.7 = y - 2$	$\frac{1}{6} + a = \frac{1}{3}$	$c + 8.1 = -6.3$
8.	$\frac{1}{2} = x + \frac{3}{4}$	$r + 2.2 = 1.8$	$4 + x = .2$

9. If 33 less than a number is -141, what is the number? Let y represent the number. Then does the sentence $y - 33 = -141$ state the given conditions? Solve the equation.

10. If 17 more than a number is -18, what is the number?

Solving Equations: The Multiplication Principle

You know that in an equation of the form $x = k$ the coefficient of x is 1. In the equations you have solved by the addition principle, the coefficient of the variable has always been 1. Now let us look for a systematic method for solving equations in which the coefficient of the variable is not 1; for example, $2x = 8$.

1. What is the root of $2x = 8$? of $3 \cdot 2x = 3 \cdot 8$? of $\frac{1}{4} \cdot 2x = \frac{1}{4} \cdot 8$? Are the three equations equivalent? Why? How were the second and third equations obtained from the first? Do you see a pattern for finding equivalent equations by the use of multiplication?

Test your answer by finding the solution sets of the following.

	a	b	c
2.	$15x = 105$	$\frac{1}{5} \cdot 15x = \frac{1}{5} \cdot 105$	$\frac{1}{15} \cdot 15x = \frac{1}{15} \cdot 105$
3.	$-\frac{1}{12}y = 8$	$-6 \cdot (-\frac{1}{12}y) = -6 \cdot 8$	$-12 \cdot (-\frac{1}{12}y) = -12 \cdot 8$
4.	$\frac{3}{8}z = 9$	$\frac{2}{3} \cdot \frac{3}{8}z = \frac{2}{3} \cdot 9$	$\frac{8}{3} \cdot \frac{3}{8}z = \frac{8}{3} \cdot 9$

Are the equations in **2** equivalent equations? in **3**? in **4**?

▶ If the members of an equation are operated on by multiplying each by the same non-zero number, an equivalent equation is obtained.

5. You know that $\frac{1}{a} \cdot a = 1$ if $a \neq 0$. Does this fact give you a clue to how you can obtain an equivalent equation such as $x = k$ from an equation in which the coefficient of the variable is not 1? Look at the equation in **2c**. How is the $\frac{1}{15}$ by which we multiplied each member related to 15, the coefficient of the variable? Complete the indicated operations. Is the equation the simplest equivalent equation of $15x = 105$?

Look at the equations in **3c** and **4c**. Are both equations of the desired form when the indicated operations are completed? How is the -12 related to the $-\frac{1}{12}$ in **3c**? How is the $\frac{8}{3}$ related to the $\frac{3}{8}$ in **4c**?

▶ If an equation of the form $ax = b$ (where $a \neq 0$) is operated on by multiplying both members by the multiplicative inverse of a—that is, $\frac{1}{a}$—the resulting equivalent equation will be of the desired form, $x = k$.

Using the multiplication principle, find the root of each of the following equations.

	a	b
6.	$25x = 425$	$\frac{3}{4}y = 42$
7.	$\frac{1}{13}g = -5$	$-14h = 49$
8.	$-\frac{1}{16}m = -8$	$\frac{4}{5}n = -100$

9. Suppose you wish to solve the equation $-3\frac{1}{6}m = -57$. What is the multiplicative inverse of $-3\frac{1}{6}$? Operate on both members of the equation with $-\frac{6}{19}$. What is the root of the original equation?

10. What is the root of the equation $-n = 17$? Can this equation be expressed as $-1 \cdot n = 17$? What is the multiplicative inverse of -1? Solve the equation.

EXERCISES

Solve each of the following equations.

	a	b	c
1.	$\frac{5}{7}r = 70$	$1\frac{3}{8}s = 88$	$7x = 3\frac{1}{2}$
2.	$-2\frac{1}{2}t = 45$	$-\frac{7}{9}u = 112$	$-5r = 5$
3.	$2.7a = 8.1$	$-\frac{1}{32}b = 2$	$9.7x = 0$
4.	$10 = \frac{4}{11}v$	$23w = 57.5$	$-3y = 2\frac{1}{4}$
5.	$-.25m = 3.78$	$-d = 2\frac{1}{9}$	$18.7 = -1.1p$
6.	$-x = -4\frac{1}{2}$	$-2 = \frac{1}{9}n$	$-\frac{4}{5}v = -25\frac{3}{5}$

7. Seven-eighths of a number is −56. Write an equation and solve it to find the number.

8. If $2\frac{1}{3}$ times a number is 100, what is the number?

Solving Equations: Using the Addition and Multiplication Principles

1. Suppose you wish to solve the equation $14x + 35 = 77$. In this case you will need to perform more than one operation to find the simplest equivalent equation, in which the root is obvious. Complete each of the following steps. Explain why each step is possible.

	$14x + 35 = 77$
Add −35 to each member:	$14x = \underline{?}$
Multiply each member by $\frac{1}{14}$:	$x = \underline{?}$

What is the root of the last equation? of the original equation?

Using a similar procedure, find the root of each of the following.

2. a. $12x + 14 = 16$ b. $\frac{1}{8}a - 4 = 12$

3. a. $\frac{2}{3}m + 20 = 0$ b. $48y + 8 = 104$

4. a. $3.2b - 2.32 = 11.44$ b. $\frac{1}{14}n - 23 = -22$

5. Do you think it makes any difference whether the addition principle is applied *before* or *after* the multiplication principle? As an experiment, solve the equation $3x + 2 = 23$ by completing the following steps.

	$3x + 2 = 23$
Add −2 to each member:	$3x = \underline{?}$
Multiply each member by $\frac{1}{3}$:	$x = \underline{?}$

What is the root of the original equation?

Now solve the equation by applying the multiplication principle first.

	$3x + 2 = 23$
Multiply each member by $\frac{1}{3}$:	$\frac{1}{3}(3x + 2) = \underline{?}$
Perform the indicated multiplication:	$x + \frac{2}{3} = \underline{?}$
Add $-\frac{2}{3}$ to each member:	$x = \underline{?}$

Do you obtain the same root as before? Which order of operations do you think is easier for solving equations of this kind?

6. In this series of equations, explain how each equivalent equation was obtained. Remember that $9 - x$ means $9 + (-x)$. Using this form of the equation explains the way in which you obtain the "$-x$" of the second equation.

$$9 - x = -10$$
$$-x = -19$$
$$x = 19$$

Find the root of each of the following. Keep in mind the fact that $b - x$ means $b + (-x)$.

a. $47.6 - d = 92.3$ b. $\frac{5}{8} = \frac{3}{4} - s$ c. $2\frac{1}{6} = 1 - t$

7. Some equations are of the form $4(x + 4) = 11$. By applying the distributive principle, you may express this equation in the form $4x + 16 = 11$. What is the root of $4(x + 4) = 11$?

Could you also obtain the root by multiplying each member by $\frac{1}{4}$ as the first step? Try it.

8. Is $\frac{x-3}{8}$ equivalent to $\frac{1}{8}(x - 3)$? Find the root of $\frac{x-3}{8} = 4$.

The equations you have been solving are of the form $ax + b = c$, where a, b, and c are rational numbers but $a \neq 0$. Does b always have an additive inverse? Why? Then you know we may write an equivalent equation of the form:

$$ax + b + (-b) = c + (-b), \text{ or } ax = c - b$$

Does a (where $a \neq 0$) always have a multiplicative inverse? Then is the equation $ax = c - b$ equivalent to:

$$\tfrac{1}{a} \cdot ax = \tfrac{1}{a}(c - b), \text{ or } x = \tfrac{1}{a}(c - b)$$

Since a, b, and c are rational numbers, is the expression $\frac{1}{a}(c - b)$ a rational number? Then $x = \frac{1}{a}(c - b)$ is an equation of the form $x = k$, where k is some number. Do you see the value of knowing that the set of rational numbers is a commutative group under addition and under multiplication (except for 0)?

EXERCISES

Find the root of each of the following equations. Again use the set of rational numbers as the replacement set.

1. $x + 5 = 8$	**2.** $8s = 30$	**3.** $\frac{1}{2} + p = -\frac{1}{2}$
4. $4 - 2y = 25$	**5.** $2x - 3 = -7$	**6.** $-3x = 12$
7. $3(y - 2) = 24$	**8.** $y - 13 = 0$	**9.** $7t = -63$
10. $-q + 2.5 = 3$	**11.** $\frac{2}{3}y = 24$	**12.** $\frac{1}{4}a + 4 = 4$
13. $7 - 3x = 8$	**14.** $2(3z + 4) = 11$	**15.** $r - 1.7 = 4.6$
16. $22 + z = 22$	**17.** $-w = 6.25$	**18.** $5(y + 3) = -5$
19. $\frac{18 - x}{4} = 3$	**20.** $3b - 7 = -28$	**21.** $\frac{5}{6}h + 1\frac{1}{6} = 2\frac{5}{6}$
22. $4(1 - x) = 12$	**23.** $\frac{2x + 5}{3} = 15$	**24.** $\frac{1}{3}(12x + 7) = 0$

Solving Inequalities: The Addition Principle

1. Before considering inequalities that are open sentences, let us consider some order relationships consisting of numbers, as follows:

$$3 < 5$$

$$3 + 4 < 5 + 4, \text{ or } 7 < 9$$

You know the first inequality is true. If 4 is added to each member, is the resulting inequality also true?

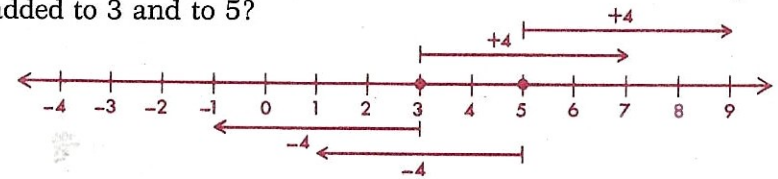
Now let us add a negative number to each member of an inequality.

$$3 < 5$$

$$3 + (-4) < 5 + (-4), \text{ or } -1 < 1$$

If -4 is added to each member of the first inequality, is the resulting inequality also true?

Look at the number line below. You see that 3 is to the left of 5. If 4 is added to 3 and to 5, are the sums in the same order; that is, is 7 to the left of 9? What happens to the order relationship if -4 is added to 3 and to 5?



You can see that 3 is less than 5, 7 is less than 9, and -1 is less than 1. Notice that the original numbers are two unit lengths apart and that each pair of sums consists of two numbers that are two unit lengths apart.

2. You know that $12 > 5$ is a true statement. Are the following inequalities also true statements?

$12 + 3 > 5 + 3$ $\qquad$ $12 + 8 > 5 + 8$

$12 + (-17) > 5 + (-17)$ $\qquad$ $12 + (-12) > 5 + (-12)$

Does it seem that if the same number, positive or negative, is added to each of a pair of numbers, the resulting pair has the same order as the original pair?

Now consider some inequalities that are open sentences. Use the replacement set $\{-3, -2, -1, 0, 1, 2, 3\}$.

3. Let us investigate the inequality $y - 3 < -2$. First, find the solution set by testing members of the replacement set. (The word *roots* is not used to name members of the solution set of an inequality.)

Now, starting with $y - 3 < -2$, add 3 to each member of the inequality. What is the solution set of $y < 1$? Is it the same set as the solution set of $y - 3 < -2$? Are the two inequalities equivalent?

$$y - 3 < -2$$
$$y - 3 + 3 < -2 + 3$$
$$\text{or } y < 1$$

4. Find the solution set of the inequality $x + 7 > 8$.

Now start with $x + 7 > 8$ and add the additive inverse of 7 to each member of the inequality. What is the solution set of $x > 1$? Are the two inequalities equivalent? Can the solution set of $x > 1$ be found by inspection?

$$x + 7 > 8$$
$$x + 7 + (-7) > 8 + (-7)$$
$$\text{or } x > 1$$

On the basis of your experience so far, do you agree with the following principle?

▶ If both members of an inequality are operated on by adding the same number to each, an equivalent inequality is obtained.

In **3–4**, in which type of inequality is the solution set obvious? An inequality having one member that consists of only the variable is called the *simplest equivalent inequality*. In solving inequalities of the form $x + b > c$, you can obtain the simplest equivalent inequality by adding the additive inverse of b to both members: $x + b + (-b) > c + (-b)$. For example, what would you add to both members of $12 + m > 8$? $d - 5 < -5$? $8.4 + x \geq -4.1$?

EXERCISES

Use the addition principle to find the solution set of each of the following inequalities.

1. $n + 73 < 75$	**2.** $m - 6.5 < -8$	**3.** $t + 2\frac{1}{4} \geq 1\frac{3}{4}$
4. $23.7 + a > 23.7$	**5.** $b + 1.75 > 0$	**6.** $s - 1.38 \leq 1.62$
7. $p - \frac{3}{4} > \frac{1}{4}$	**8.** $2\frac{1}{2} + r < 1\frac{1}{4}$	**9.** $11 - x \geq 17$
10. $7 \geq x + 2$	**11.** $y + \frac{3}{2} < 1$	**12.** $a - 19 > -2$
13. $s - 5 < -5$	**14.** $t + 4 \leq 1$	**15.** $6.7 + m < 7.8$

Solving Inequalities: The Multiplication Principles

1. Examine the order relationships in the following inequalities.

$$5 < 9$$
$$4 \cdot 5 < 4 \cdot 9, \text{ or } 20 < 36$$

You know that 5 is less than 9. Is 20 less than 36?

2. Start with another inequality you know to be true.

$$-8 < -3$$
$$5 \cdot (-8) < 5 \cdot (-3), \text{ or } -40 < -15$$

If each member of the inequality is multiplied by 5, is the resulting inequality also true?

3. Check two more examples of the same kind.

a. $24 > 12$
$3 \cdot 24 > 3 \cdot 12$

b. $10 > -10$
$7 \cdot 10 > 7 \cdot (-10)$

Are both inequalities true in **a**? in **b**? Does it seem that if each of a pair of numbers is multiplied by the same *positive* number the resulting pair has the same order as the original pair?

4. Now, starting with an inequality that is true, multiply each of a pair of numbers by the same negative number.

$$2 < 4$$
$$-3 \cdot 2 < -3 \cdot 4, \text{ or } -6 < -12$$

Is the resulting inequality a true statement; that is, is -6 less than -12? See whether you obtain a true statement if you reverse the order, or direction, of the second inequality; that is, change "$<$" to "$>$."

$$2 < 4$$
$$-3 \cdot 2 > -3 \cdot 4, \text{ or } -6 > -12$$

Is -6 greater than -12?

5. You know that the inequality $-7 < -2$ is a true statement. Multiply each member by -7 and change the inequality symbol from "$<$" to "$>$."

$$-7 < -2$$
$$-7 \cdot (-7) > -7 \cdot (-2), \text{ or } 49 > 14$$

Is the resulting inequality a true statement?

6. Notice that in the following examples, as in **4** and **5**, the inequality symbol is reversed in the second inequality.

a. $8 > 2$
$-9 \cdot 8 < -9 \cdot 2$

b. $13 > -17$
$-4 \cdot 13 < -4 \cdot (-17)$

Are both inequalities true in **a**? in **b**?

Does it seem that if each of a pair of numbers is multiplied by the same *negative* number the resulting pair is in reverse order from the original pair?

The principles that you have found to be true of inequalities consisting only of numbers may now be tested for inequalities that are open sentences. Use the replacement set $\{-3, -2, -1, 0, 1, 2, 3\}$ in the following work.

7. Suppose you wish to solve the inequality $5x < 10$. First, find the solution set by testing different numbers from the replacement set. Then, starting with $5x < 10$, multiply each member of the inequality by $\frac{1}{5}$. What is the solution set of $x < 2$? Is the solution set the same as that of $5x < 10$? Are the two inequalities equivalent?

$$5x < 10$$
$$\tfrac{1}{5} \cdot 5x < \tfrac{1}{5} \cdot 10$$
$$\text{or } x < 2$$

In which inequality is the solution set obvious? How is the multiplier $\frac{1}{5}$ related to the coefficient of x?

8. Suppose you wish to solve the inequality $\frac{1}{3}y < 1$. First, find the solution set by trying different replacements. Then, starting with $\frac{1}{3}y < 1$, multiply each member of the inequality by 3. What is the solution set of $y < 3$? Is this the same as the solution set of $\frac{1}{3}y < 1$? Are $\frac{1}{3}y < 1$ and $y < 3$ equivalent?

$$\tfrac{1}{3}y < 1$$
$$3 \cdot \tfrac{1}{3}y < 3 \cdot 1$$
$$\text{or } y < 3$$

9. What is the solution set of each of the following inequalities? Are they equivalent?

$$8s > 12$$
$$\tfrac{1}{8} \cdot 8s > \tfrac{1}{8} \cdot 12, \text{ or } s > 1\tfrac{1}{2}$$

10. What is the solution set of each of the following inequalities? Are they equivalent?

$$\tfrac{1}{4}t > -\tfrac{7}{16}$$
$$4 \cdot \tfrac{1}{4}t > 4 \cdot (-\tfrac{7}{16}), \text{ or } t > -1\tfrac{3}{4}$$

In doing **7–10**, have you found that the following principle is true?

▶ If both members of an inequality are operated on by multiplying each by the same *positive* number, an equivalent inequality is obtained.

Do you notice that in each exercise the second inequality has a coefficient of 1 for the variable? Describe the relationship between the multiplier in the second inequality and the coefficient of the variable in the first. For example, in **8**, how is the 3 related to the $\frac{1}{3}$?

11. Look at the inequality at the right. What is the solution set of $x > -2$? Is this also the solution set of $-2x < 4$? Test each member of the replacement set to be sure. Are $-2x < 4$ and $x > -2$ equivalent inequalities?

$$-2x < 4$$
$$-\tfrac{1}{2} \cdot (-2x) > -\tfrac{1}{2} \cdot 4$$
$$\text{or } x > -2$$

12. Examine the steps taken in the example at the right. What is the solution set of $y > 2$? Is this also the solution set of $-\frac{1}{6}y < -\frac{1}{3}$? Are the two inequalities equivalent? In which inequality is the solution set obvious?

$$-\tfrac{1}{6}y < -\tfrac{1}{3}$$
$$-6 \cdot (-\tfrac{1}{6}y) > -6 \cdot (-\tfrac{1}{3})$$
$$\text{or } y > 2$$

13. What is the solution set of each of the following inequalities? Are they equivalent? Is the symbol reversed in the second inequality?

$$-11n > -3$$
$$-\tfrac{1}{11} \cdot (-11n) < -\tfrac{1}{11} \cdot (-3), \text{ or } n < \tfrac{3}{11}$$

14. What is the solution set of each of the following inequalities? Are they equivalent? Is the symbol reversed in the second inequality?

$$-\tfrac{3}{5}w > -1\tfrac{4}{5}$$
$$-\tfrac{5}{3} \cdot (-\tfrac{3}{5}w) < -\tfrac{5}{3} \cdot (-\tfrac{9}{5}), \text{ or } w < 3$$

Do you think the following statement is true?

▶ If both members of an inequality are operated on by multiplying each by the same *negative* number, an equivalent inequality is obtained by changing "<" to ">" or by changing ">" to "<."

Do you agree that the simplest equivalent inequality is obtained if each member is multiplied by the multiplicative inverse of the coefficient of the variable?

EXERCISES

Use the multiplication principles to find the solution set of each of the following inequalities.

	a	b	c
1.	$5m < -7$	$-8h < 5$	$-3 < -y$
2.	$-2.2y > -4.84$	$-x < 1.25$	$4a \geq 0$
3.	$\frac{2}{3}x < -1$	$-\frac{1}{8}k < \frac{1}{4}$	$\frac{2}{3}x < 1$
4.	$\frac{5}{6}w \geq 1\frac{2}{3}$	$-\frac{3}{4}r \leq 1\frac{3}{4}$	$-2m < -14$
5.	$-\frac{2}{5}c \leq 10$	$1.6p > 2.56$	$\frac{8}{9}n \leq -72$
6.	$4y \geq -3$	$3x \leq -51$	$-.2a \leq 2.2$

Solving Inequalities: Using Two Operations

You can now solve more complicated inequalities by using both the addition and multiplication principles that you have studied in the preceding sections. To obtain a simple equivalent inequality, first apply the addition principle, if necessary, and then the multiplication principle, if necessary. In the following work, use the set of rational numbers as the replacement set.

1. To solve the following inequality:

$$12x - 57 > 51$$

Add 57 to each member:

$$12x - 57 + 57 > 51 + 57, \text{ or } 12x > 108$$

Now multiply each member by $\frac{1}{12}$:

$$\tfrac{1}{12} \cdot 12x > \tfrac{1}{12} \cdot 108, \text{ or } x > 9$$

Do you see that the solution set of the original inequality is all numbers greater than 9? This may be expressed as $\{x \mid x > 9\}$.

2. To solve the following inequality:

$$14 - 9w < 59$$

Add the additive inverse of 14 to each member:

$$14 - 9w + (-14) < 59 + (-14), \text{ or } -9w < 45$$

Now multiply each member by $-\frac{1}{9}$. Is the direction of the inequality reversed?

$$-\tfrac{1}{9} \cdot (-9w) > -\tfrac{1}{9} \cdot 45, \text{ or } w > -5$$

What is the solution set of $14 - 9w < 59$?

Solve the following inequalities.

3. **a.** $23t + 77 < 146$ **b.** $-19n + 95 > -95$
4. **a.** $\frac{1}{5}h - 39.8 > -34.7$ **b.** $1.5k - 1.3 < 3.2$
5. **a.** $-\frac{1}{4}m + \frac{1}{2} > \frac{3}{4}$ **b.** $\frac{3}{5} - \frac{2}{5}r < \frac{4}{5}$
6. **a.** $1 + \frac{2}{3}s \leq -1$ **b.** $-\frac{3}{4} - \frac{1}{2}e \leq 1\frac{3}{4}$

EXERCISES

Using the principles you have learned, find the one inequality under **a**, **b**, **c**, or **d** that is equivalent to the original inequality. You do not need to give the solution set.

	a	b	c	d
1. $-5m > 10$	$m > 2$	$m < 2$	$m > -2$	$m < -2$
2. $5 + n < 11$	$n > 6$	$n < 6$	$n > 16$	$n < -16$
3. $5 + 2n > 11$	$n > 3$	$n < 3$	$n > 8$	$n < -8$
4. $5 - 2n < 11$	$n > 3$	$n < 3$	$n > -3$	$n < -3$
5. $2n - 5 > 11$	$n > 8$	$n < 8$	$n > -8$	$n < -8$
6. $3 - x > 5$	$x > 8$	$x < 8$	$x < -2$	$x > -2$
7. $7 - 3x < -8$	$x > 5$	$x < 5$	$x > -5$	$x < -5$

Find the solution set of each of the following inequalities. Use the set of rational numbers as the replacement set.

8. $x-4<7$ **9.** $8g<4$ **10.** $2x-7<0$
11. $5-5p>25$ **12.** $\frac{1}{2}m>4.5$ **13.** $r+\frac{1}{4}>\frac{3}{4}$
14. $3(k+1)<4$ **15.** $y+17<20$ **16.** $4b+8\leq16$
17. $-2h<-12$ **18.** $2(2x+1)>7$ **19.** $s-2.3\geq3.4$
20. $\frac{1}{3}q+\frac{2}{3}>1$ **21.** $-3n>15$ **22.** $5(5-m)\leq0$
23. $\frac{1}{2}(\frac{1}{2}+m)>1\frac{1}{2}$ **24.** $-9+w\leq9$ **25.** $-\frac{3}{4}w>\frac{4}{5}$
26. $3r\leq-7$ **27.** $4(-3a-4)<-16$ **28.** $15-\frac{2}{3}y>21$
29. $1\frac{2}{3}+t>0$ **30.** $-3c-2<1$ **31.** $-3(\frac{1}{3}d+9)>-24$

PROBLEM SOLVING

Number Problems

1. Ask a friend to choose a number, multiply it by 6, add 15, and tell you the result. By using an equation, you can tell your friend what number he originally chose.

Suppose the result is 39. If you let x represent the original number, would $6x + 15$ represent the result after multiplying the number by 6 and adding 15? Since 39 is also the result, you can write the equation $6x + 15 = 39$. Then the solution set of this equation is the original number. What is the original number? As a check, multiply the number by 6 and then add 15. Is the result 39?

2. Think of a number, multiply it by 2, and add 7. The result is 17. What is the number?

3. Think of a number, multiply it by 5, and add 11. The result is $28\frac{1}{2}$. What is the number?

4. Think of a number, multiply it by 8, and subtract 40. The result is -56. What is the number?

5. If you subtract a number from 33, the result is 18. What is the number?

6. Think of a number and add 7. Then multiply the sum by 4. The result is 44. What is the number?

7. If you add 13 to a number and then divide by 2, the result is $77\frac{1}{2}$. What is the number?

8. A store advertises a blanket sale in which prices are one-third less than the original price. If a blanket is now selling for \$13.32, what was its original price?

Problems Involving Formulas

A formula is usually in the form of an equation showing the relationship between two or more variables. If you know the replacements for all of the variables but one, you can find the other by making the replacements in the equation and solving the equation.

1. One formula that describes the relation between Centigrade (or Celsius) temperature readings and Fahrenheit readings may be written $\frac{9}{5}C + 32 = F$. If you know the number of Fahrenheit degrees (F), you can find the corresponding number of Centigrade degrees (C). Suppose the temperature is 68° Fahrenheit. You can write the equation as $\frac{9}{5}C + 32 = 68$. Solve the equation to find the corresponding temperature on the Centigrade scale.

2. Using the same formula, find the Centigrade temperature reading that corresponds to each of the following Fahrenheit readings.

a. 95°F **b.** 17°F **c.** 0°F **d.** 40°F **e.** −4°F

3. As you know, the formula for finding the circumference of a circle is $c = 2\pi r$, where c is the number of units of length in the circumference and r is the number of units of length in the radius. You may use $\frac{22}{7}$ as an approximate value of π. If the radius of a circle is 28 feet, what is the circumference of the circle?

4. Suppose you wish to find the radius of a circle when you know the circumference. If the circumference of a circle is 132 feet, what is the radius? If the circumference is 352 feet, what is the radius?

5. The formula for finding the area enclosed by a circle is $A = \pi r^2$, where A is the number of square units in the area and r is the number of units of length in the radius. If the radius of a circle is 10 inches, what is the area?

Inequality Problems

In each of the following problems, choose a variable to represent the desired number, write an inequality describing the information given, and find the solution set.

1. A certain number plus 3 is less than 12. The number is less than ?.

2. Twice the population of the United States is greater than 350,000,000. The population is greater than ?.

3. Bill deposited \$12 in his savings account. He now has more than \$37 in his account. How can you describe the amount of his account before the deposit?

4. If three times a certain number is subtracted from 100, the result is less than 40. How can you describe the original number?

LIKE TERMS IN OPEN SENTENCES

Defining Like Terms

In working with equations and inequalities, we use the word **term.** In mathematics, a term is any of the following:

1. A number, such as 15 or $88\frac{1}{2}$
2. A variable, such as x or n
3. The product of two terms, of a term and an expression, or of two expressions; for example, $5x$, xy, $5(a + b)$, $(a + b)(c + d)$

4. The quotient of two terms, of a term and an expression, or of two expressions; for example, $\frac{a}{b}$, $\frac{a+b}{c}$, $\frac{c}{a+b}$, $\frac{a+b}{c+d}$. In a term that is a quotient, the divisor cannot be zero.

You will notice that every term is also an expression, but not every expression is a term. For example, $5a + 5b$ is an expression with two terms.

A term that is a number, such as 15 or $88\frac{1}{2}$, is called a **constant.**

An expression may have more than one term containing the same variable; for example, $3x + 4x$. If two or more terms have the same variable or variables, and the corresponding variables have the same exponent, the terms are called similar terms, or **like terms.**

Which of the following expressions contain like terms?

1. a. $14a - 5a$ **b.** $3x + 4y$

2. a. $3n + 4n^2$ **b.** $6c^2 - 5c^2$

3. a. $5t - 4t + 2t$ **b.** $x + y + z$

Since the distributive principle applies to multiplication with respect to addition, can we write the expression $9x + 16x$ as $(9 + 16)x$? Can $(9 + 16)x$ also be expressed as $25x$? Can we also write the expression $9x - 16x$ as $(9 - 16)x$? as $-7x$? You can see that in an expression containing like terms, the distributive principle allows us to perform the indicated addition or subtraction. Thus, we can write an expression containing only like terms as a single term.

Write each of the following expressions as a single term.

4. a. $39y + 17y$ **b.** $-27r + 45r$

5. a. $8.3s - 3.8s$ **b.** $16.2k - 23.4k$

6. a. $4\frac{1}{2}d + d + \frac{1}{2}d$ **b.** $\frac{1}{3}n + 2\frac{1}{3}n - n$

When the variable in an expression is replaced by a number, you know that we say we have *evaluated the expression* at that number. Suppose we have the expression $9x - 7x$, and we use 5 as a replacement for x. Do you see that the value of the expression is $(9 \cdot 5) - (7 \cdot 5)$, or $45 - 35$, or 10? We have *evaluated the expression at 5.* We can also say that 10 is *the value of the expression at 5.* Of course, we can evaluate the expression at any number in the replacement set. What is the result if $9x - 7x$ is evaluated at 8?

Like Terms in Equations

Sometimes an equation contains like terms in one or both of its members. Expressing the like terms in each member as a single term is the first step in solving such an equation.

You know that it is also necessary to perform any indicated operations with numbers in order to find the simplest equivalent equation.

One member of each equation below contains like terms. In each case, express the like terms as a single term. Then find the root of the equation.

	a	b
1.	$4x + 2x = 8$	$3.7x + 6.3x = 57 + 41$
2.	$5x - 2x = 15$	$3\frac{1}{2}x - 3\frac{1}{4}x = 18 - 20$
3.	$14x - 15x = 2\frac{1}{2}$	$27x - 36x = 3(4 + 2)$

Some equations contain like terms in both members. To obtain an equivalent equation with the variable in only one member, a term containing the variable is added to each member of the equation. The term chosen to be added is such that after simplification the variable appears in only one member.

Suppose you wish to solve the equation: $7x = 9 + 4x$

Add the additive inverse of $4x$ to each member: $7x + (-4x) = 9 + 4x + (-4x)$

Express like terms as a single term: $3x = 9$

Multiply each member by $\frac{1}{3}$: $x = 3$

What is the root of the original equation?

Solve each of the following equations.

	a	b
4.	$9x = 2x + 21$	$8x - x = 24 - 5x$
5.	$2x + 35 = 7x$	$3x + 4x - 42 = 14x$
6.	$-10.5x = 18 - 11.5x$	$10x = 2x - 12 + 3x$

Perform any indicated multiplication and find the root of each of the following equations.

	a	b
7.	$-5x = 3(x + 15)$	$2(2x - 4) = x$
8.	$\frac{3}{8}x = \frac{1}{2}(\frac{3}{8}x + \frac{1}{8})$	$\frac{2}{3}x = \frac{2}{3}(\frac{3}{4}x - 30)$
9.	$(2.2x + 3.6)2.5 = 8.5x$	$(1.1x - 1.1)1.1 = .21x$

Sometimes an equation contains like terms and constants in each member. To obtain the simplest equivalent equation, it may be necessary to use all the steps you have practiced in solving equations.

As an illustration, consider the equation:

$$\frac{1}{3}(x - 4) + 3\frac{1}{3} = 3\frac{2}{3} - \frac{1}{6}(2x - 2)$$

Apply the distributive principle:

$$\frac{1}{3}x - 1\frac{1}{3} + 3\frac{1}{3} = 3\frac{2}{3} - \frac{1}{3}x + \frac{1}{3}$$

Simplify: $\frac{1}{3}x + 2 = 4 - \frac{1}{3}x$

Add $\frac{1}{3}x$ to each member: $\frac{2}{3}x + 2 = 4$

Add -2 to each member: $\frac{2}{3}x = 2$

Multiply each member by $\frac{3}{2}$: $x = 3$

What is the root of the original equation?

EXERCISES

Find the root of each of the following equations.

1.	a. $17 - x = 5 + 2x$	b. $x + 3 - 4x = 3(x - 4)$
2.	a. $3x - 4 = 2x - 4$	b. $\frac{3}{4}(x - 4) = 3x + 1$
3.	a. $3x - 4 = 8 - x$	b. $2x - 3 = \frac{1}{3}(9 - 6x) - 6$
4.	a. $15x - 12 = 3x + 24$	b. $7(x - 3) = x - 21$
5.	a. $\frac{1}{3}m + 2 = m - 4$	b. $4(x - 1) = 7 - 3 - 4x$

6. $\frac{1}{5}(y - 10) + 16 = \frac{3}{5}y + 17 - \frac{1}{5}y$
7. $5x - \frac{1}{2}(12 + 10x) = \frac{3}{4}(6 + 4x)$
8. $\frac{1}{2}(4x - 9) - 11\frac{1}{2} = 17 + (-2)(1\frac{1}{2}x + 4)$

Like Terms in Inequalities

Working with like terms in inequalities is similar to working with like terms in equations. Like terms in one member of an inequality may be expressed as a single term. If like terms occur in both members, the same term is added to each member. The term chosen for adding is such that in one member the term with the variable is eliminated. The inequality is then solved by the methods you have already studied.

For example, find the solution set of:

$$3x - 5 > 7x + 8$$

Add the additive inverse of $7x$ to each member. Why?

$$3x - 5 + (-7x) > 7x + 8 + (-7x)$$

Express the like terms in each member as single terms.

$$-4x - 5 > 8$$

Add 5 to each member. Why?

$$-4x > 13$$

Multiply each member by $-\frac{1}{4}$. Why?

$$x < -3\tfrac{1}{4}$$

The solution set is the set of all numbers less than $-3\frac{1}{4}$; that is, $\{x \mid x < -3\frac{1}{4}\}$.

EXERCISES

Find the solution set of each of the following inequalities.

	a	b
1.	$4y - 7 < 2y + 4$	$h - 3 > 3h + 9$
2.	$3x - 9 < 2 + 4x$	$3 - 2n < n - 5$
3.	$5 - 7x \geq 3 + x$	$4(x - 3) < 3x + 5$
4.	$.2(x - 5) \leq 4.2x + 1$	$3x \geq x$
5.	$\frac{x}{2} \geq x$	$.2(x - 5) \leq 4.2x - 1$

Solve each of the following equations and inequalities. The replacement set is the set of rational numbers.

6.	$5a + 4a = 126$	$2b - 5b = 1\frac{1}{2}$
7.	$-7h + 6h = 12$	$5k - 8 + 2k = 14$
8.	$7c - c = 26 - 7c$	$\frac{1}{2}d + 5\frac{1}{2}d = 18 + 2d$
9.	$18x - 10x < 80$	$-3y + 15y > 30$
10.	$1\frac{1}{2}n + 1\frac{1}{2}n < -10$	$7m - 12m > 40$
11.	$5s + s < s - 3.5$	$-11t + 3t > 11 - 4t$
12.	$16r + 24 = 19r + 33$	$18 - 12w = 12w - 9$
13.	$\frac{3}{4}n + 7\frac{1}{2} = 2\frac{1}{2} + 1\frac{3}{4}n$	$2.3 - 3.2m = 3.2 - 2.3m$
14.	$3(8 - 2y) = 4 + y$	$3(u - 9) = 5(3 + u)$
15.	$3g - 4 < 6g + 2$	$-12 + 5h > 10 + 3h$
16.	$11 + 2p < 15p + 1$	$13q + 19 > 59 + q$
17.	$3(5z - 7) > 4(-8 + 2z)$	$\frac{2}{3}(\frac{1}{2}x - \frac{9}{10}) < 2\frac{1}{3}x + \frac{2}{5}$

PROBLEM SOLVING

Using Equations and Inequalities

In each of the following problems, let a variable represent some unknown number. The problem itself tells you what the replacement set is. Check your solution in the original problem to make sure that it satisfies all of the conditions.

1. If you multiply a certain number by 4 and subtract 16, the result is the number with which you started. What is the number?

2. If you multiply a certain number by 3 and add 7, you will get a result that is 11 less than 5 times the original number. What is the number?

3. The number of people attending a P.T.A. meeting was 78. There were 5 times as many parents as teachers at the meeting. How many teachers attended the meeting? how many parents?

4. Multiplying a number by 7 and then adding 18 gives the same result as multiplying the number by 18 and adding 7. What is the number?

5. Adding 12 to a number gives the same result as dividing it by 7. What is the number?

6. Five years from now Jack will be 10 years less than twice as old as he is now. How old is he now?

7. Four times a number is greater than the number added to 15. The number is greater than ?.

8. Two times a certain number is less than the number. Describe the number.

9. Bill is 3 years younger than Fred. The sum of their ages is greater than 25. Fred is more than ? years old.

10. Julie bought three dresses, each for the same price, and a fourth dress for twice as much as each of the three. The total cost was less than \$45. How can you describe the cost of each dress?

11. Mr. Morgan bought twice the number of rosebushes that he had in his garden, but had to throw 3 away. When the new bushes were planted, he had more than 48 rosebushes in his garden. How can you describe the number of rosebushes he had originally?

Number Problems and Relationships

1. If you designate the first of three consecutive integers by x, how can you represent the second? If the second is represented by $x + 1$, how can the third be represented? Does the equation $x + (x + 1) + (x + 2) = 21$ tell you that the sum of three consecutive integers is 21? Solve the equation. What are the integers?

2. The sum of four consecutive odd integers is 80. What are the integers?

Let x stand for the first integer. How much larger is the next odd integer? Is it represented by $x + 2$? How can you represent the third odd integer? the fourth? Now you can write the following equation.

$$x + (x + 2) + (x + 4) + (x + 6) = 80$$

What is the replacement set for x? Solve the equation. What are the four integers? Is their sum 80? If the root of the equation turned out to be an even integer, what would the solution set be?

3. The sum of four consecutive even integers is 76. What are the integers?

4. The sum of five consecutive odd integers is 205. List them.

5. The sum of five consecutive integers is 205. List them.

6. Are 4, 5, and 6 consecutive integers? What is the sum of 4, 5, and 6? Is the sum divisible by 3?

Are 22, 23, and 24 three consecutive integers? What is their sum? Is the sum divisible by 3?

Try another set of three consecutive integers. Is their sum divisible by 3? Do you think the sum of *any* three consecutive integers is divisible by 3? Let x represent the first integer. Then how can you represent the second and third integers? Does the following expression represent the sum of the three integers?

$$x + (x + 1) + (x + 2)$$

Simplify this expression.

$$x + x + 1 + x + 2 = ?$$

Can $3x + 3$ be expressed as $3(x + 1)$? You know that x is an integer. Then for any replacement of x by an integer, will $x + 1$ be an integer? So $3x + 3$ can be expressed as the product of 3 and some integer. Then the sum of any three consecutive integers has 3 as a factor. Can it therefore be divided by 3?

7. If x stands for the first integer, can the sum of the four consecutive integers be represented as $x + (x + 1) + (x + 2) + (x + 3)$? Simplifying, we obtain $4x + \underline{?}$ as the sum.

If $4x + 6$ is to be divisible by 4, we must be able to express it as a product in which one factor is 4 and the other represents an integer.

$$4x + 6 = 4(x + \underline{?})$$

Since the replacement set for x is the set of integers, can $x + 1\frac{1}{2}$ represent an integer? Thus, the sum of four consecutive integers cannot be divisible by 4.

Do the following steps clearly show your work?

$$x + (x + 1) + (x + 2) + (x + 3) = 4x + 6$$
$$= 4(x + 1\tfrac{1}{2})$$

8. Use the same procedure to discover whether the sum of any five consecutive integers is divisible by 5.

9. Is the sum of any six consecutive integers divisible by 6?

10. In a certain two-digit numeral, the units' place digit is twice the tens' place digit. The number it names is 18 more than the sum of the digits. What is the number?

Let x represent the tens' place digit. Then how would you represent the units' place digit? Can you express the number as $10x + 2x$? Why? What is the sum of the digits? Then does the following equation express the information given in the problem?

$$10x + 2x = x + 2x + 18$$

Solve the equation. What is the tens' place digit? the units' place digit? What is the number?

11. The units' place digit of a two-digit number is 4 more than the tens' place digit. The number is 7 more than 3 times the sum of its digits. What is the number?

12. The tens' place digit of a two-digit number is twice the units' place digit. If you reverse the digits, the resulting number is 27 less than the original number. What is the original number? (First write an expression that represents the original number. Then write an expression that represents a number with the same digits reversed. You are then ready to write an equation.)

Coin and Mixture Problems

1. Two children have been collecting dimes and nickels by selling lemonade and cookies. They have 22 coins that total \$1.45. How many dimes do they have? how many nickels?

Let x stand for the number of dimes. Is the value of the x dimes $10x$ cents? If x is the number of dimes and there are 22 coins altogether, how many nickels are there? What is the value in cents of $22 - x$ nickels? Is the total value of the coins in cents $10x + 5(22 - x)$? Is the total value also 145 cents? Do you see that you can now write the equation:

$$10x + 5(22 - x) = 145$$

Solve the equation to find the number of dimes. Then what is the number of nickels?

2. The 27 coins in a piggy bank are worth \$4.20. If it is known that all of the coins are quarters or dimes, how many of each kind of coin are in the bank?

3. In a container there are twice as many pennies as nickels and the same number of dimes as nickels. If the value of the coins is \$1.19, how many of each kind of coin are there in the container?

4. In a container there are pennies, nickels, dimes, and quarters. There are twice as many nickels as pennies, three times as many dimes as nickels, and half as many quarters as pennies. The total number of coins is 133. How many of each kind of coin are there in the container? What is the total value?

5. A 10% antifreeze solution is one that contains .10 antifreeze and .90 water. How much antifreeze must you add to 5 quarts of a 10% antifreeze solution in order to get a 20% antifreeze solution?

Let x stand for the amount of antifreeze to be added. Could you express the amount of antifreeze in 5 quarts of a 10% solution as $.10(5)$? Then how would you express the amount of antifreeze in $(5 + x)$ quarts of a 20% solution? You know that the amount of antifreeze in the original solution plus the amount of antifreeze to be added will equal the amount of antifreeze in the new solution. Therefore, you can write the equation:

$$.10(5) + x = .20(5 + x)$$

Solve the equation to find the amount of antifreeze to be added.

6. How much antifreeze must you add to 5 quarts of a 15% antifreeze solution in order to get a 25% antifreeze solution?

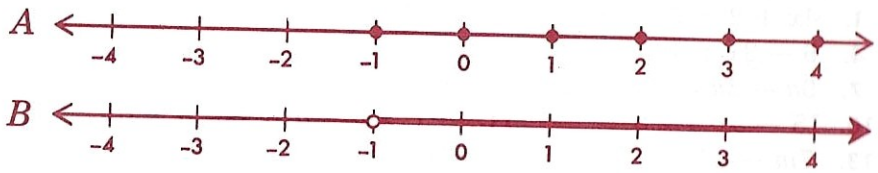
7. How many gallons of maple syrup must be added to 20 gallons of a mixture of 40% maple syrup and 60% cane-sugar syrup in order to get a mixture with 50% maple syrup?

GRAPHS OF SOLUTION SETS

Equations and Inequalities

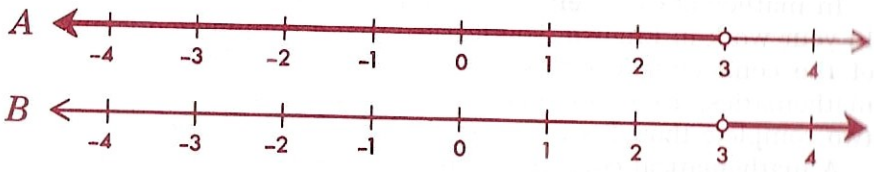
In Chapter 1 you saw how a number line can be used to give a picture, or graph, of the solution set of an open sentence such as $x + 3 > 4$. At that time, the replacement set for the variable was the set of numbers of arithmetic. Let us now investigate the graphs of solution sets of sentences where the replacement set is the set of rational numbers.

1. Suppose you wish a graph of the solution set of the sentence $3y > -3$. Can the solution set be defined as $\{y \mid y > -1\}$? Which of the two graphs below is the graph of the solution set?



In graph B, why is there a hollow dot at the point corresponding to -1? What does the arrow extending to the right indicate?

2. Find the solution set of the sentence $4 > x + 1$. Which of the graphs below is the graph of the solution set?



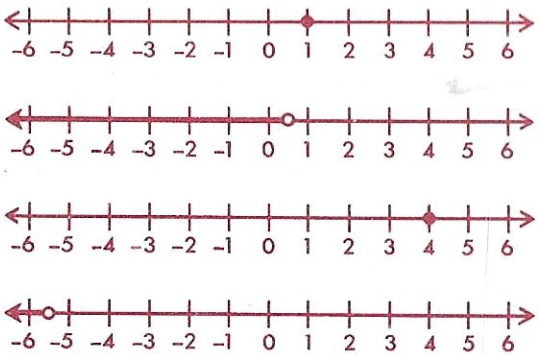
3. Below are four sentences. Find the solution set of each and match it with one of the graphs labeled A, B, C, and D.

a. $3r - 6 = 6$

b. $2x + 3 = 5$

c. $y + 3 < -2\frac{1}{2}$

d. $x - 3 < -2\frac{1}{2}$



4. Draw a graph of the solution set of $-5 < x$.

5. Draw a graph of the solution set of $x < -8$.

EXERCISES

Draw graphs of the solution sets of the following equations and inequalities.

1. $4x + 2 = 2x + 2$	**2.** $2x + 1 = 7$	**3.** $4 - 5x = -11$
4. $5 - 3x > 14$	**5.** $y + 3 < 4$	**6.** $3x - \frac{1}{2} = 4$
7. $9a - 3a < 18a - 12$	**8.** $4x > 2x + 1$	**9.** $2a - 1 < 5$
10. $15 - 5x = 2.5$	**11.** $-1 > 2 - b$	**12.** $4 - 3x < -2$
13. $7m - 14 > -7$	**14.** $4x + 8 > 8$	**15.** $\frac{1}{5}t - 1 < 0$

Conjunction

Look at the sentence at the right. This sentence presents two complete thoughts (that is, it is a combination of two sentences) connected by the word *and*.

$8 > 6 + 1$ *and* $2 + 3 = 5$

In mathematics, a sentence of this kind is called a **conjunction.** In your work in English, you learned that a conjunction is any one of the connecting words such as "and," "but," "for," etc. But in mathematics, a conjunction is a mathematical sentence in which two complete thoughts are connected by the word *and*.

A mathematical conjunction is true if both of its parts are true. If either of its parts is false, we say that the conjunction is false. Is the conjunction above true, or is it false?

1. Which of the following conjunctions are true and which are false?

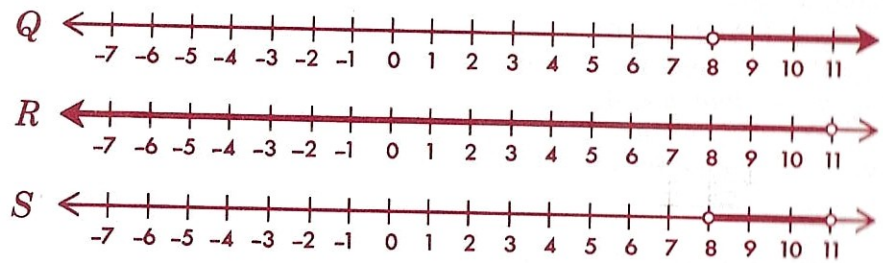
a. $7 > 4 + 2$ *and* $5 < 7$
b. $5 + 1 < 5$ *and* $6 - 1 = 5$
c. $2 = 3 - 1$ *and* $3 > 2$
d. $4 = 5 + 1$ *and* $2 < 4$
e. $1 - 1 < 0$ *and* $4 - 3 > 1$
f. $6 < 8$ *and* $3 - 1 = 4$

2. At the right is an open sentence in the form of a conjunction. The solution set of this sentence consists of only those replacements for y that will make both parts true statements.

$8 < y$ *and* $y < 11$

Is 15 a member of the solution set? Why? Is 7 a member of the solution set? Why? Is 10 a member of the solution set? Why?

3. Graphs are often helpful in obtaining the solution set of a conjunction. You can draw the graph of the solution set of each part and then note the points that are common to both graphs. The numbers corresponding to these points are the members of the solution set of the conjunction. Below we have drawn three graphs. What does graph Q represent? graph R? Do you agree that the graph labeled S represents the graph of the solution set of the sentence $8 < y$ *and* $y < 11$?



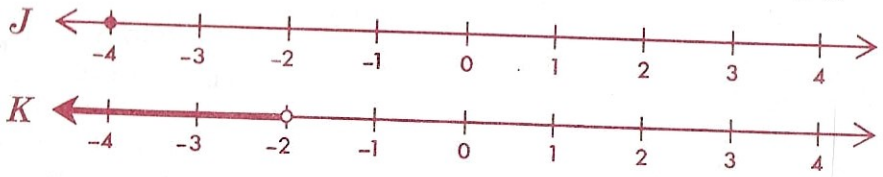
Why is 8 not a member of the solution set? Why is 13 not a member of the solution set?

Actually, in this particular case, you might have arrived at the solution set easily by recalling your work in Chapter 2. There you found that if a number a is greater than a number b and less than a number c, the number a is between b and c. Thus, $8 < y$ *and* $y < 11$ is equivalent to $8 < y < 11$, and the solution set consists of all numbers between 8 and 11.

Study the definition of betweenness on page 84. Do you see that betweenness is defined in terms of a conjunction?

4. Draw a graph of the solution set of $-\frac{1}{2} < q$ *and* $q < \frac{1}{2}$.

5. Use graphs to find the solution set of the sentence $x < -2$ *and* $x = -4$. Graph J, below, is the graph of the solution set of which part of the conjunction? graph K? What number (or numbers) corresponds to the only point (or points) common to the two graphs? Is $\{-4\}$ the solution set of the sentence $x < -2$ *and* $x = -4$?



6. Determine the solution set of the sentence $z < 1$ *and* $z > \frac{1}{2}$.

7. Consider the sentence $r > 7$ *and* $r < 5$. Is there any replacement for r that will make both parts true statements? Then the solution set of the sentence is $\emptyset$. Show this by using graphs.

EXERCISES

Draw the graphs of the solution sets of the following conjunctions. Below each graph, describe or tabulate the solution set of the conjunction.

1. $y > 1$ *and* $y < 6$
2. $a = 3$ *and* $a < 5$
3. $x > -1$ *and* $x < 3$
4. $t < 0$ *and* $t = -2$
5. $y \geq 2$ *and* $y < 4$
6. $x = 1$ *and* $x < -1$
7. $a = -2$ *and* $a = 0$
8. $m > 5$ *and* $m < 3$
9. $m < 5$ *and* $m > 3$
10. $x > -2$ *and* $x < 0$

Disjunction

Look at the sentence at the right. Here we have a sentence with two parts connected by the word *or*. In mathematics, a sentence of this kind is called a **disjunction.**

$2 + 9 = 11$ *or* $1\frac{1}{2} + \frac{1}{4} > 2$

A disjunction is true if *either* of the parts is true; it is also true if *both* of the parts are true. If neither of the parts is true, the disjunction is false. Is the disjunction above true, or is it false?

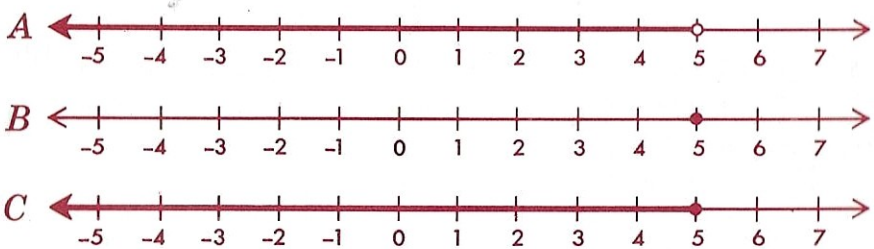
1. Which of the following disjunctions are true, and which are false?

a. $3 = 2 + 1$ *or* $5 - 3 = 2$
b. $3 < 2$ *or* $6 > 8$
c. $4 < 5$ *or* $7 = 5 + 1$
d. $6 = 3 + 2$ *or* $2 < 3$
e. $4 < 6$ *or* $7 > 6$
f. $8 = 5 + 3$ *or* $9 < 7$

2. As you can see, the open sentence at the right has the form of a disjunction. Any number that makes at least one part true will make the disjunction true. Thus, the solution set of the sentence will consist of all numbers that make either part a true statement. Will 6 satisfy at least one of the parts? will $5\frac{1}{2}$? 5? 4? $3\frac{1}{2}$? 3.1? -2? Do you see that 5 and every number less than 5 will be a member of the solution set?

$x < 5$ *or* $x = 5$

We can also use graphs to illustrate the solution set of a disjunction. Below we have drawn a graph (A) of the solution set of $x < 5$, and a graph (B) of the solution set of $x = 5$. The combined graph (C) represents the graph of the solution set of the sentence $x < 5$ *or* $x = 5$. Is the disjunction $x < 5$ *or* $x = 5$ equivalent to $x \leq 5$?



3. Draw the graph of the solution set of the sentence $x > -3$ *or* $x = -3$. Write another sentence that expresses this disjunction.

4. You know that the graph of the solution set of a conjunction consists of all the points that satisfy *both* parts of the conjunction. Do you see that the graph of the solution set of a disjunction consists of any points that satisfy *either* part of the disjunction? For example, describe the graph of the solution set of the conjunction $x = 2$ *and* $x = 3$; the solution set of the disjunction $x = 2$ *or* $x = 3$.

5. a. Look at the sentence $m = 8$ *or* $m = -8$. Tabulate the solution set and draw the graph of the solution set.

b. Consider the sentence $|\, m \,| = 8$. This sentence, you remember, is read "the absolute value of m is equal to 8." How is absolute value defined? (Refer to page 55 to refresh your memory.)

Will we obtain a true statement if m is replaced by -8? if m is replaced by 8? Then the solution set of $|\, m \,| = 8$ is $\{-8, 8\}$.

Does $|\, m \,| = 8$ have the same solution set as the disjunction $m = 8$ *or* $m = -8$? Are they equivalent sentences?

6. Tabulate the solution set of $|\, y \,| = 1$.

EXERCISES

Draw the graphs of the solution sets of these conjunctions and disjunctions. Below each graph, tabulate or describe the solution set that is pictured.

1. $y > 1$ *and* $y < 6$
2. $a = 4$ *and* $a > 0$
3. $2x < 12$ *and* $x < 5$
4. $y < 1$ *or* $y > 6$
5. $a = 4$ *or* $a > 0$
6. $a > 0$ *and* $a < -1$
7. $y = 7$ *and* $y < 8$
8. $y < 3$ *or* $y > 2$
9. $x \geq 2$ *or* $x < -2$
10. $a \leq \frac{1}{2}$ *and* $a \geq \frac{1}{2}$
11. $s = 3$ *and* $3 \leq s$
12. $m + 2 > 4$ *and* $m = 5$
13. $x \leq 4$ *or* $x > 4$
14. $1 - y = 2$ *and* $y < 0$
15. $x = 4$ *and* $x = |-4|$
16. $2a > a + 2$ *and* $a = 3$
17. $-7 < t$ *and* $-1 > t$
18. $x = -3$ *or* $x = |-3|$
19. $\frac{1}{2}x < 4$ *and* $x > \frac{1}{2}$
20. $y < -2$ *or* $y > 2$

1. Using the replacement set $\{1, 2, 3, 4\}$, find the number pairs that make up the solution set of each of the following open sentences. List the pairs in the same order as the letters appear in each sentence. 90

a	b	c
$x + y = 4$	$c - d = 2$	$r - t = -3$

Use the set of rational numbers as the replacement set for **2–12.** Under **a**, **b**, and **c**, find the two open sentences that are equivalent. 91–92

2.	$n + 3 = 22$	$n + 3 = 16$	$n = 19$
3.	$6y = -18$	$2y = -8$	$8y = -32$
4.	$s - 3 < 0$	$s - 6 < -3$	$s - 9 < 6$

Find the root of each of the following equations. 94–96

5.	$x - 18 = 38$	$m + \frac{3}{8} = \frac{3}{4}$	$a - 1.4 = 3.3$
6.	$13h = 65$	$\frac{1}{7}w = 10$	$\frac{3}{5}c = -21$
7.	$4y - 11 = 9$	$3n + 4 = 6$	$\frac{1}{5}x - \frac{2}{5} = \frac{2}{5}$

Find the solution set of each of the following inequalities. 99, 102–103

8.	$n - 11 < 22$	$y + 3\frac{1}{2} \geq 5$	$7k < 40$
9.	$-5x > 55$	$3a - 4.3 \leq 2.7$	$15 - 8d > 71$

Solve each of the following open sentences. 108–110

	a	b
10.	$6w + 8w = 42$	$6n - 9n + 6 = -15$
11.	$4x - 3 = 2x + 9$	$3y + 11 = 7y + 13$
12.	$5x + 7x > 36$	$4x + 17 < 9x + 27$

13. If a certain number is multiplied by 7, and 8 is added to the result, the total is 43. What is the number? 104–105

14. The sum of three consecutive integers is 123. What are they? 112–113

15. Describe or tabulate the solution sets of the following conjunctions and disjunctions. 116–119

a. $x > 2$ *and* $x < 4$
b. $a \leq 3$ *and* $a = 1$
c. $|x| = 5$
d. $y < 6$ *or* $y \leq 4$

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 399

Challenge Page 414

4 Polynomials

PROPERTIES

Description In Chapter 3 you used algebraic expressions to solve various types of problems. In order to proceed more deeply into the structure and uses of algebra, it is necessary to study the properties of a special set of algebraic expressions known as polynomials.

We can begin by defining the simplest form of a polynomial, a monomial.

▶ A **monomial** is any one of the following:

1. A number—for example, 7, $\frac{1}{2}$, 103
2. A variable—for example, x, y, t
3. The product of numbers, variables, or numbers and variables —for example, $5x$, $.3a$, $\frac{1}{4}pq$, a^2

Does this definition of monomials remind you of the definition of a term? There is an important difference, however. Look back at pages 106–107. Is the expression $\frac{x}{y}$ a term? It is not, however, a monomial, since it is the *quotient* of two variables, not the product. Is $\frac{x}{2}$ a term? May it be expressed as $\frac{1}{2}x$? Is it a polynomial?

1. Express each of the following as a monomial in simplest form.

a. $x \cdot x$ **b.** $\frac{2 \times 4}{2}$ **c.** $-1 \cdot y \cdot y \cdot 3$

d. $1 \cdot m \cdot n \cdot n$ **e.** $3 \cdot \frac{1}{2} \cdot a \cdot b$ **f.** $5 \cdot c \cdot c \cdot 2 \cdot d$

▶ A **polynomial** is a monomial or the sum of monomials.

A member of the set of polynomials may have any number of terms, provided each term is a monomial. Although *poly* means "many," the word *polynomial* has come to mean an expression with one or more terms.

2. This is a polynomial:

$$3x^2 + 5x + 7$$

These are polynomials:

$2x$ $\qquad$ $-b + 4$

$\frac{x}{3} + 2$ $\qquad$ $x^3 - 2x^2 - 7x$

$2y^3 + 3y - 8 - 6y^2$ $\qquad$ $27a^2 + 8a - 164$

These are not polynomials:

$\frac{1}{x} + 2$ $\qquad$ $y^2 + \frac{3}{y} + 4$ $\qquad$ $\frac{1 - 3}{y}$

Notice that the sign of operation between terms may be a minus sign. Is $4a - 1$ equal to $4a + (-1)$? Do you see that $\frac{x}{3}$ may be written as the product $\frac{1}{3}x$, but that even if we write $\frac{3}{y}$ as $3 \cdot \frac{1}{y}$ there is still a variable in the denominator and the term is an indicated quotient?

3. Decide which of the following are polynomials and which are not. If any are not, explain why they are not.

a. $2x^2 + 2x + 2$ **b.** $\frac{2x + 7}{x} + 9x$

c. $16u + 98u^2$ **d.** $19x^3 - 40 + 274x^2$

e. $7z^4 + z^3 - z^2 + 4z - 3$ **f.** $2x^2 + 3x - 7 + \frac{5}{x}$

g. $\frac{1}{4}a$ **h.** $\frac{x + 4}{2}$

Shown below is the algebraic expression that defines the form of a polynomial in one variable; in this case, a polynomial in x.

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + F$$

(Where the capital letters represent constants and all the exponents are positive integers.)

This algebraic expression is the **standard form of a polynomial.** Notice that the variable is never in the denominator of a fraction. The terms are arranged in descending order of exponents. Any expression that is in this standard form or can be changed to this form is a member of the set of polynomials.

4. Let us examine the following expression to see whether it is a polynomial.

$$2x^2 + 3x^4 - 2 + x$$

Rearranged with a descending order of exponents, we have:

$$3x^4 + 2x^2 + x - 2$$

Can we write this as follows?

$$3 \cdot x^4 + 0 \cdot x^3 + 2 \cdot x^2 + 1 \cdot x^1 + (-2)$$

Now compare this expression with the algebraic expression for the standard form of a polynomial. What has replaced A? x^n? B? x^{n-1}? C? x^{n-2}? D? x^{n-3}? F? Is the expression a polynomial?

5. Examine your answers to **3** and put into standard polynomial form the expressions that are polynomials. Which polynomials have terms with zero coefficients or zero as a constant?

6. One method of classifying polynomials is by the number of terms they have. You know that 3, x, $27y^3$, and $\frac{1}{2}x^2$ are called *monomials* because each possesses just one term. The polynomial $7x + 4$ is called a **binomial** because it has two terms; $3x^3 + 4x^2 - 9$ is called a **trinomial** because it has three terms. When a polynomial has more than three terms, we do not usually give it a special name. Look at the expressions in **3** that are polynomials and identify those that are monomials; binomials; trinomials.

Evaluating Polynomials

1. You know that when performing operations with numbers or variables there is a definite order to be followed. Sometimes the order is described by the use of mathematical punctuation, such as parentheses and brackets. In this book we use mathematical punctuation. However, at other times you may have to remember the proper order without the aid of punctuation. Study the steps followed in simplifying the expression below and describe the type or types of operations performed at each step.

(1) $2 \cdot 3^2 + 4 \div 2 - 3 \cdot 2^2 + 6^2$
(2) $2 \cdot 9 + 4 \div 2 - 3 \cdot 4 + 36$
(3) $18 + 2 - 12 + 36$
(4) 44

We may summarize the order of operations as follows. Perform each operation from left to right before moving on to the next step.

1. Find any powers that are indicated.
2. Perform any multiplication or division necessary.
3. Perform any addition or subtraction necessary.

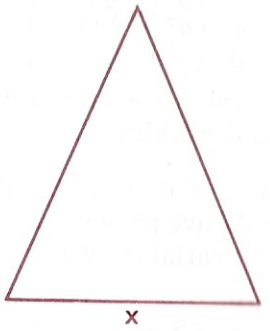
2. In Chapter 1 you learned that making a replacement for the variable in an expression is called "evaluating" the expression. To evaluate the polynomial $x^2 + 3x + 7$, the variable is replaced by a number. Evaluate this polynomial at each of the following.

a. 2 **b.** 0 **c.** -3 **d.** 1 **e.** 5 **f.** $\frac{1}{2}$

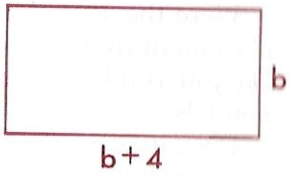
3. Evaluate $x^3 - 2x^2 + 4x - 10$ at each of the following.

a. 2 **b.** 0 **c.** -1 **d.** 10

4. In this isosceles triangle, the length of the base is x inches. The sides with equal measures are each 2 inches longer than the base. Represent each of these sides as a polynomial in x. May we then represent the perimeter of the triangle in inches by the polynomial $3x + 4$? Evaluate the polynomial at 4; at 12. What is the perimeter of the triangle when the base is $6\frac{1}{2}$ inches? 8.5 inches?



5. Represent the perimeter in feet of this rectangle as a polynomial having b as the variable. Find the perimeter when b is replaced by 1; 6; $3\frac{1}{2}$; 3.2.



Principles

1. Are 2 and 3 integers? Is the sum of 2 and 3 also an integer? Is the sum of any two integers an integer? What principle does this illustrate?

Is x a polynomial? Is 3? Is $x + 3$ a polynomial? Write a polynomial that is the sum of each of the following pairs of polynomials.

a. 7 and x **b.** -7 and x
c. 2 and $3x$ **d.** $(x - 1)$ and $(x + 3)$
e. $(y^2 + 4)$ and $(y^3 + 1)$ **f.** $(a^2 + b^2)$ and $(a^2 - b^2)$

Does it seem reasonable to assume that the set of polynomials is closed under addition?

2. For any particular replacement for x, does $x + 3$ name the same number as $3 + x$; that is, does $x + 3 = 3 + x$? Test your answer by evaluating each member at 7; -2; $\frac{1}{3}$. Do you think $3x^2 + 4x = 4x + 3x^2$? $y^4 + y^2 = y^2 + y^4$? $(x + 1) + x^2 = x^2 + (x + 1)$? Try a few replacements for x and for y if you have any doubt. What is each of the terms in the sentence? What principle does this illustrate for polynomials under addition?

3. Look at these sentences. Replace the variables by 3; -2; 10. Are the sentences true for each replacement? Do you think the sentences would be true for all replacements for the variables?

a. $x + (2 + 3) = (x + 2) + 3$
b. $(y + 2y) + 4 = y + (2y + 4)$
c. $(a^2 + 2a) + (-7) = a^2 + (2a + (-7))$
d. $(x + 3) + 4x^2 = x + (3 + 4x^2)$

Does it seem that the associative principle applies to polynomials under addition?

4. Let us examine the set of polynomials to see whether the distributive principle holds for this set. In each sentence below, replace the variable by 1; 8; $\frac{2}{3}$; -5.

a. $3(a + 2) = 3a + 6$ **b.** $x(x - 4) = x^2 - 4x$
c. $(5 + 1)x = 5x + x$ **d.** $7x - \frac{1}{2}x = (7 - \frac{1}{2})x$

Were the sentences true for each replacement? Try to find a replacement that will make even one of the sentences false. Can you? Do you think the distributive principle holds for the set of polynomials?

Does it seem that the set of polynomials under the operation of addition has the principles of closure, commutativity, and associativity? Does the distributive principle for multiplication over addition also apply? We shall assume that these principles do hold true for the set of polynomials under addition.

Using Polynomials in Problems

1. If an object is dropped and allowed "free fall" (air resistance neglected), it is found that it will fall approximately 16 feet in the first second, 64 feet in two seconds, 144 feet in three seconds, 256 feet in four seconds, 400 feet in five seconds, and so forth. How far does it fall during the third second? during the fourth second?

Is 16 a factor of each of the numbers 16, 64, 144, 256, and 400? Is 16 equal to 16×1^2? Is 64 equal to 16×2^2? How is 16 related to 144? 256? 400? Can you write the formula that will describe the distance in feet (s) an object falls in a given number of seconds (t)? Does the formula $s = 16t^2$ fit the requirements?

2. If an object is dropped in free fall and hits the ground at the end of 8 seconds, from what height was it dropped?

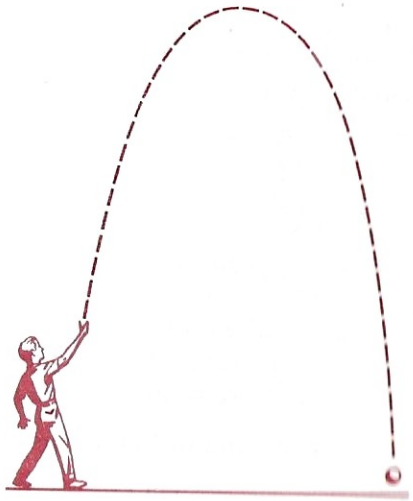
3. A boy dropped a rock off a cliff and counted 6 seconds before he heard the rock hit. The cliff must be about ? feet high.

4. The observation platform on a certain cliff is 1024 feet above the base of the cliff. A ball is dropped in free fall from the top of the cliff. How many seconds will elapse before the ball hits the ground below?

5. An object is dropped from a height of 64 feet. Remember that the distance it falls is $16t^2$. Can we express its height at the end of t seconds with the polynomial $64 - 16t^2$?

Evaluate the polynomial at 0; at $\frac{1}{2}$; at 1; at $1\frac{1}{2}$; at 2. Does this polynomial have a reasonable meaning for t replacements greater than 2? Why?

6. If an object is thrown upward, the path that it follows (up and down) is called its **trajectory.** It is possible to develop a polynomial that will describe the height of the object above the ground at any given time for any given force of throw. For example, if a ball is thrown upward at a speed of 64 feet per second, the polynomial that describes its height t seconds after it is thrown is $64t - 16t^2$. Evaluate the polynomial at 4. How do you explain your answer? Does the polynomial tell us that the ball has fallen to the ground at the end of 4 seconds?



7. Evaluate $64t - 16t^2$ at 0. Explain your answer.

Find the value of the polynomial at 1; at 2; at 3. What do you think a reasonable replacement set is for t?

Find the value at 2.1; at 1.9. At what value of t do you think the ball achieves its greatest height?

8. In a certain office building there are 100 telephone extensions. It is possible to dial from any extension to any other extension. Do you see that only 50 telephone connections are possible at any one time in this system? However, there are many more possible connections between the telephones. If e represents the number of extensions, the polynomial $\frac{1}{2}e(e-1)$ describes the number of possible connections. How many different connections are possible in the office building with 100 extensions?

9. Use the formula $c = \frac{1}{2}e(e-1)$ to find how many connections are possible in a telephone system with 3000 telephones. How many connections are possible at one time?

ADDITION AND SUBTRACTION

Combining Like Terms

1. You know that in performing addition with integers we consider place value. Look at the examples below. Do you see that we have actually treated the numbers as polynomials and applied the distributive principle?

$$\begin{array}{ll} 521 & 5 \cdot 10^2 + 2 \cdot 10 + 1 \\ 456 & 4 \cdot 10^2 + 5 \cdot 10 + 6 \\ \hline 977 & (5+4)10^2 + (2+5)10 + (1+6), \text{ or } 977 \end{array}$$

Now suppose that instead of adding two numbers expressed as powers of ten we wish to add two polynomials in x. You know that we can combine only terms that contain the same variables with the same powers; that is, we can combine only like terms. Are $4x^3$ and $7x^3$ like terms? Why? Are $2x^2$ and $2x$ like terms? Why not?

2. Choose pairs of like terms from the following sets of terms.

a. $3x,\ 5x^3,\ 6x^2,\ y^4,\ -x^2,\ 2x$

b. $3x,\ 2y^2,\ -4x^2,\ -4y,\ y^3,\ -4y^2$

Addition

1. Study the following addition of $(3x^2 + 5x + 7)$ and $(2x + 1)$. The sum can be written as:

$$(3x^2 + 5x + 7) + (2x + 1)$$

What principles allow this to be written as:

$$3x^2 + (5x + 2x) + (7 + 1)$$

What principle allows this expression to be written as:

$$3x^2 + (5 + 2)x + (7 + 1)$$

Then the standard form of the sum is:

$$3x^2 + 7x + 8$$

▶ The addition of polynomials is performed by writing the indicated sum, applying the commutative and associative principles to collect like terms, applying the distributive principle, and then writing the terms in descending order of exponents.

2. Copy the following addition example. After each step, describe what was done and what principle or principles permit us to do it.

$$\begin{aligned} (x^2 + 4x + 2) + (2x^2 + 5x + 3) &= x^2 + 4x + 2 + 2x^2 + 5x + 3 \\ &= x^2 + 2x^2 + 4x + 5x + 2 + 3 \\ &= (1+2)x^2 + (4+5)x + (2+3) \\ &= 3x^2 + 9x + 5 \end{aligned}$$

For convenience, addition of polynomials is frequently performed vertically. Are the principles that allow the above example as obvious in the vertical example? Do they still apply? Do you see that like terms are written one under another, just as in arithmetic hundreds are written under hundreds, and so on?

$$\begin{array}{r} x^2 + 4x + 2 \\ 2x^2 + 5x + 3 \\ \hline 3x^2 + 9x + 5 \end{array}$$

3. Now consider the addition of the two polynomials $x^2 - 5x + 7$ and $2x - 5 + 3x^2$.

$$\begin{aligned}(x^2 - 5x + 7) + (2x - 5 + 3x^2) &= x^2 - 5x + 7 + 2x - 5 + 3x^2 \\ &= x^2 + 3x^2 - 5x + 2x + 7 - 5 \\ &= (1 + 3)x^2 + (-5 + 2)x + (7 - 5) \\ &= 4x^2 - 3x + 2\end{aligned}$$

Arranging the same addition example in a vertical form, we have:

$$\begin{array}{r} x^2 - 5x + 7 \\ 3x^2 + 2x - 5 \\ \hline 4x^2 - 3x + 2 \end{array}$$

Why did we rearrange the terms of the second expression? If both expressions were already in standard polynomial form, would they need to be rearranged? Is x^2 equal to $1 \cdot x^2$? Is $x^2 - 5x$ equal to $x^2 + (-5x)$?

EXERCISES

Add.

1. $\begin{array}{r} 2x^2 + 7x + 9 \\ 5x^2 + x + 11 \\ \hline \end{array}$

2. $\begin{array}{r} 2x^2 + 15x + 19 \\ x^2 + 7x - 10 \\ \hline \end{array}$

3. $\begin{array}{r} 12u^2 + 7u - 11 \\ 5u^2 + 2u - 3 \\ \hline \end{array}$

4. $\begin{array}{r} 6x^2 - 4x - 3 \\ x^2 - 3x - 1 \\ \hline \end{array}$

5. $\begin{array}{r} y^2 - 3y - 8 \\ y^2 + 3y + 9 \\ \hline \end{array}$

6. $\begin{array}{r} 2x^3 - 3x^2 + 9x \\ 7x^3 + x^2 + x \\ \hline \end{array}$

7. $\begin{array}{r} 42z^3 + 19z^2 - 67z + 265 \\ 12z^3 - 15z^2 - 124z - 807 \\ \hline \end{array}$

8. $\begin{array}{r} 9x^3 - 3x + 2 \\ 2x^2 + 4x - 7 \\ \hline \end{array}$

9. $9x^2 + 14x - 10$ and $2x^2 - 7x + 4$

10. $y^2 + 8y - 11$ and $7y^2 - 8y - 9$

11. $1.3x - 9.6$ and $7.2x + 15.5$

12. $18z^3 + 9z^2 + 4z - 1$ and $2z^3 + 4$

13. $2y^3 - 8y + y^2 + 2$ and $3y - 2y^2 - 2 + y^3$

14. $7x - 9x^2 + 4$ and $x^2 + 17x^3 - 9 + 4x$

15. Add the sum in Exercise **1** to the sum in Exercise **2**.

Add.

16. $4x^3 - 3x + 2$ and $7 - 7x^2 - 2x - x^3$

17. $\frac{3}{5}x + 9$ and $\frac{1}{3}x - 2$

18. $1.7y - 1.4y^3$ and $.8y^2 + .7$

19. $\frac{1}{2}y^3 + \frac{1}{3}y - \frac{2}{5}$ and $2y^2 + 3 - \frac{2}{3}y$

20. $\begin{array}{r} x^3 + 2x^2 - x + 7 \\ 9x^2 + 2x - 1 \\ \hline 3x^3 - 4x^2 + 5x + 2 \end{array}$

21. $\begin{array}{r} x^3 - 19x^2 + 2x + 1 \\ x^2 - 3x - 9 \\ 2x^3 - 3x + 4 \\ \hline -x^3 - 4x^2 + 7x \end{array}$

The Identity Polynomial and the Additive Inverse

1. What is the identity element for addition in the set of integers? in the set of rationals? When this identity element is added to a number, what is the sum? What can be added to the polynomial $x + 2$ so that the sum will be $x + 2$? Is 0 a polynomial? What polynomial can be added to $2x^2 - 3x + 4$ so that the sum will be $2x^2 - 3x + 4$? Do you think 0 is the identity element for addition in the set of polynomials?

2. What is meant by the "additive inverse" of a number in the set of rational numbers? In the set of integers, what is the additive inverse of 3; that is, $3 + \underline{?} = 0$? What is the additive inverse of -6? of 0? of $\frac{2}{7}$?

3. Let us look for the additive inverses of polynomials. For instance, is there a polynomial which when added to $x + 2$ produces 0? Complete these four addition examples to help you find the answer.

a. $(x + 2) + (2 + x) = ?$ **b.** $(x + 2) + (-x + 2) = ?$

c. $(x + 2) + (x - 2) = ?$ **d.** $(x + 2) + (-(x + 2)) = ?$

What is the additive inverse of $x + 2$? Can it be expressed as $-x - 2$?

4. What is the additive inverse of the polynomial $3x - 5$; that is, $(3x - 5) + \underline{?} = 0$?

5. Add $-x^2 + 2x - 3$ to $x^2 - 2x + 3$. What is the sum? Is $-x^2 + 2x - 3$ the additive inverse of $x^2 - 2x + 3$?

Is $x^2 - 2x + 3$ the additive inverse of $-x^2 + 2x - 3$? Does $-x^2 + 2x - 3$ equal $-(x^2 - 2x + 3)$?

6. Write the additive inverse of each of the following.

a. $2x^2 - 7$ b. $x^3 + 7x^2 - x - 4$

c. $y^3 + 2y^2 - 10$ d. $ax^2 - bx + c$

e. $-3x^2 + \frac{1}{2}x - 1$ f. $27z - 80z^3 + 111z^2 - 9$

7. How can you check your answers in 6?

8. What polynomial is its own additive inverse?

Polynomials as a Commutative Group

Do the polynomials constitute a well-defined set? Describe the set. Let us examine our findings about polynomials under addition.

1. Is the set of polynomials closed under addition? Give two examples to support your answer.
2. Is addition of polynomials commutative? Give an example.
3. Is addition of polynomials associative? Give an example.
4. Is there an identity element for addition in the set of polynomials? What is it?
5. Does every polynomial have an additive inverse that is a polynomial? How is the inverse related to the original polynomial?

You know that the principles covered by the questions above are the properties of a commutative group. Do you think the set of polynomials is a commutative group under addition? Have you found any instances in which any of the five principles do not hold for polynomials? Does it seem reasonable to you that they always hold? We shall assume that they are true for the set of polynomials under addition.

Subtraction

1. You know that subtraction is the inverse of addition. To subtract is to add the additive inverse of the subtrahend to the minuend. Study the following pairs of examples to refresh your memory.

a. $8 - 5 = 3$
$8 + (-5) = 3$

b. $8 - (-5) = 13$
$8 + 5 = 13$

c. $-3 - 5 = -8$
$-3 + (-5) = -8$

d. $-3 - (-5) = 2$
$-3 + 5 = 2$

Explain how in each case the subtraction example was changed to an addition example.

2. You know how to add polynomials. Does every polynomial have an additive inverse? Do you think we can subtract polynomials as we subtract numbers?

Let us try by subtracting $2x + 3$ from $3x - 5$. What is the additive inverse of $2x + 3$? The example can be expressed as follows:

Subtraction example	*Transformed to addition example*
$3x - 5$	$3x - 5$
$2x + 3$	$-2x - 3$
	$x - 8$

To check the answer, add $x - 8$ to $2x + 3$. Is the sum the original minuend?

3. Subtract $x^3 - 5x^2 - 4$ from $3x^3 - 5x^2 + 4x$. Will the following addition example give the correct answer? Check the result.

$$\begin{array}{r} 3x^3 - 5x^2 + 4x + 0 \\ -x^3 + 5x^2 + 0x + 4 \\ \hline 2x^3 \qquad + 4x + 4 \end{array}$$

4. Subtract $x^2 + 6x + 1$ from $2x^2 - 7x + 2$. Check your answer.

5. Subtract $7y^2 + 2y - 1$ from $3y^2 - 8y + 1$, using the horizontal form. Check.

EXERCISES

1. Copy the following examples and subtract.

a. $\begin{array}{r} 7y + 11 \\ 2y - 2 \\ \hline \end{array}$ b. $\begin{array}{r} 6x^2 + 5x - 2 \\ 2x^2 - 5x - 2 \\ \hline \end{array}$ c. $\begin{array}{r} 14x^2 + 2x + 18 \\ 3x^2 + 6x - 4 \\ \hline \end{array}$

2. From $4z^2 - 5z - 2$ subtract $2z^2 + 2z + 2$.

3. Subtract $9x^2 - 2x + 7$ from $20x^3 - x^2 + 4x$.

4. Copy the following examples and subtract. If necessary, rearrange the terms.

a. $\begin{array}{r} x^2 + 2 \\ x - 2 \\ \hline \end{array}$ b. $\begin{array}{r} x^2 + x \\ x + 1 \\ \hline \end{array}$ c. $\begin{array}{r} -1 + 2z - 3z^2 + z^3 \\ 3z^3 - 3z^2 - 2z + 1 \\ \hline \end{array}$

5. Subtract $3 - 3x + 5x^2$ from $2x^2 - 2x - 7$.

6. Subtract $x^2 + .3$ from $3.1x^3 + 2x$.

7. $(4x - 5x^2 + 3x^3) - (x^3 - 5x^2 - 4) = ?$

8. Subtract $x - 3$ from the sum of $6x - 2$ and $x - 7$.

9. From $3x - 4$ subtract the sum of $2x + 1$ and $2x - 5$.

10. Subtract $3x^2 + x - 5$ from the sum of $6x^2 + 3x - 2$ and $2x^2 - 6x + 1$. Will the example at the right yield the correct answer? Explain the third addend.

$$\begin{array}{r} 6x^2 + 3x - 2 \\ 2x^2 - 6x + 1 \\ \underline{-3x^2 - \quad x + 5} \end{array}$$

Arrange the following in column form as in Exercise **10** and perform the required operations.

11. $(x + 2) + (3x - 3) - (2x + 1) = ?$
12. $(a - 2) - (2a + 3) + (a + 6) = ?$
13. $(a - 2) + (a + 6) - (2a + 3) = ?$
14. $(a - 2) + (2a + 3) - (a + 6) = ?$
15. $(x^2 - 6x - 2) + (3x - 1) - (x^2 + 5) = ?$
16. $(\frac{1}{2}y + \frac{1}{3}) - (\frac{1}{3}y - \frac{2}{3}) + (\frac{1}{6}y - \frac{1}{3}) = ?$
17. $(x^2 + x - 1) - (2x - 4) - (x^2 - 1) = ?$

Without using the column form, perform the following operations.

18. $(3x + 2) + (2x - 3) - (5x - 1) = ?$
19. $(x^2 - 4x + 1) - (5x + 2) - (x^2 + 3) = ?$
20. $(a + 2) + (a - 2) - (2 - a) - (-a - 2) = ?$
21. $(x^2 - 2x + 4) - (2x^2 + 3) + (8x + 7) - (2x^2 - x + 19) = ?$

On Your Own

Using a procedure similar to the one used on pages 112–113 in Chapter 3, construct a proof that shows:

The sum of seven consecutive integers is (or is not) divisible by 7.

The sum of eight consecutive integers is (or is not) divisible by 8.

The sum of nine consecutive integers is (or is not) divisible by 9.

The sum of ten consecutive integers is (or is not) divisible by 10.

Look for a pattern in this work and the work on pages 112–113. Do you think the sum of eleven consecutive integers is divisible by 11? the sum of twelve consecutive integers is divisible by 12? Do you think that the sum of an odd number of consecutive integers is always divisible by that number?

1. Write the letter of each expression that is a polynomial. 122–124

a. 0 b. x c. $2(x + 2)$

d. $-y^4 - \frac{y}{3}$ e. $x + \frac{1}{x}$ f. $a + a^2 - \frac{a + 2}{a}$

2. Write the following polynomials in standard form. 123–124

a. $x - 2x^2$ b. $4 + t^2 - 2t$ c. $\frac{a}{2}$

3. Evaluate $y^3 + 4y^2 - 2y + 1$ at each of the following. 125

a. 1 b. 4 c. -2

4. A flowerpot is dropped in free fall from a window 64 feet above the ground. How many seconds will it take to hit the ground? (Remember, $s = 16t^2$.) 126–127

5. Add. 129–130

a. $\begin{array}{r} 4x^2 - 2x + 1 \\ \underline{x^2 - \quad x + 1} \end{array}$ b. $\begin{array}{r} 3a^3 - 5a^2 \\ \underline{5a^3} \end{array}$ c. $\begin{array}{r} y^4 + 2y^2 - 3 \\ \underline{2y^4 + 3y^2 + 8} \end{array}$

6. Add the following pairs of polynomials. Give your answers in the standard form. 129–130

a. $b^3 - b$ and $b^2 - b^3$

b. $2x^3 - x^2 + 8x - 1$ and $3x + 4x^3$

c. $7 - 3x + 4x^2$ and $6 + 5x - 8x^2$

7. What is the identity element for polynomials under addition? 131

8. Give the additive inverse of each of the following polynomials. 131–132

a. $4x^2 + x - 3$ b. $-6a^2 - 4a + \frac{1}{2}$

9. Subtract. 132–133

a. $\begin{array}{r} x^3 + 2x - 1 \\ \underline{3x^3 + 4x + 5} \end{array}$ b. $\begin{array}{r} y^4 - y^2 \\ \underline{-y^4 - y^2} \end{array}$ c. $\begin{array}{r} 3 - x^2 + x \\ \underline{7 + 2x^2 + 4x} \end{array}$

10. Subtract $x^2 + 3x + 2$ from the sum of $4x^2 - 3x - 6$ and $-3x^2 + 5x + 8$. 134

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 4(

Challeng Page 4

MULTIPLYING AND FACTORING POLYNOMIALS

Monomials

You have already had practice in finding the product of polynomials in the special case of a monomial times a monomial. To refresh your memory, supply the answers to the following examples. If you have trouble, review pages 32–33.

1. a. $2x \cdot 3x$ **b.** $3x^2 \cdot 7x^2$ **c.** $-3a \cdot 2a$

d. $2s^2 \cdot 3s$ **e.** $4y \cdot 2y^3$ **f.** $-x \cdot (-4x^2)$

g. $2x \cdot 3x^2$ **h.** $8u \cdot \frac{1}{2}u$ **i.** $2ax^2 \cdot 3abx^3$

j. $3x^2 \cdot (-2x) \cdot \frac{1}{2}x^3$ **k.** $-x^2 \cdot 4x \cdot \frac{3}{5}x^3$ **l.** $2x^3 \cdot 5x \cdot (-3a^2)$

2. Is each of the answers to **1a**, **b**, and **c** a monomial? Do you think the set of monomials is closed under multiplication?

Now interchange the monomials in **1d**, in **1e**, and in **1f** and multiply. What other principle seems to apply to multiplication with monomials?

Next, repeat **1j**, **k**, and **l**, but this time find the product of the second and third factors first and then multiply this product by the first factor in the example. Does it seem that the associative principle also applies to multiplication with monomials?

3. You know that $a^3 \cdot a^3 = a^6$. Could we express $a^3 \cdot a^3$ as $(a^3)^2$? Then what does $(a^3)^2$ equal? Test your answer by replacing a by an integer in the two expressions.

How many times is a^2 used as a factor in $(a^2)^3$? Is $a^2 \cdot a^2 \cdot a^2$ equal to a^6? Then what is $(a^2)^3$ equal to? Test your answer.

A monomial such as $(a^3)^2$ is a power of a power. Do you see how it differs from $a^3 \cdot a^2$, which equals a^5? If a is replaced by 2, what is the value of $(a^3)^2$? of $a^3 \cdot a^2$?

You have learned that the following principles hold for exponents, where m and n are non-negative integers.

(1) $a^m \cdot a^n = a^{m+n}$

(2) $a^m \div a^n = a^{m-n}$, where $a \neq 0$ and $m \geq n$

(3) $a^0 = 1$, where $a \neq 0$

To this list we may now add:

(4) $(a^m)^n = a^{mn}$

4. Express each of the following as a power of the base.

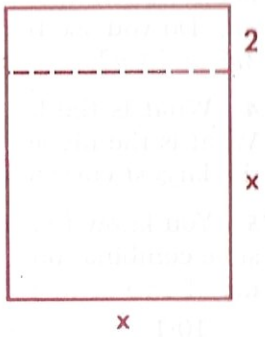
a. $(b^3)^4$ **b.** $(x^4)^2$ **c.** $(a^3)^3$

d. $(x^4)^3$ **e.** $(y^3)^6$ **f.** $(b^6)^3$

5. Does $(b^5)^2$ equal $(b^2)^5$? Does it seem that $(a^m)^n$ and $(a^n)^m$ are equal? Test your answer by replacing a, m, and n by positive integers.

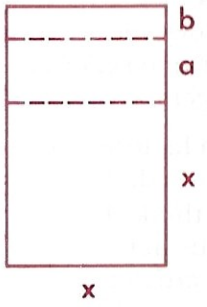
Monomials and Other Polynomials

1. Study the rectangle at the right. The height, divided into two parts, is $x + 2$ units. The base is x units. The area enclosed by the rectangle is then the product of the base and the height, or $x(x + 2)$. Notice that the area of the rectangle has been divided into two non-overlapping sections. Is x^2 the area of the lower section? What is the area of the upper section? Then the total area enclosed is $x^2 + 2x$. Do you see that $x(x + 2) = x^2 + 2x$? What principle does this illustrate?



2. By using a rectangle with base x and height $x + a + b$, show that $x(x + a + b) = x^2 + ax + bx$. Do you think that finding the product of a monomial and a polynomial always obeys the generalized distributive principle?

$$a(b + c + d) = ab + ac + ad$$



Multiply.

3. a. $x(2x + 3) = ?$ **b.** $2x(x^2 - 4x + 2) = ?$

4. a. $m(m^2 - 2m) = ?$ **b.** $(2x + 3)x = ?$

5. a. $x^2(x + 2) = ?$ **b.** $5p^2(2p^2 - 9p + 1) = ?$

6. a. $2k^2(3k - 7) = ?$ **b.** $-3x(2x - 3x^2 - 7) = ?$

Common Monomial Factors

To **factor** a polynomial is to write a given polynomial as an indicated product of two or more polynomials. Factoring polynomials can greatly simplify the solution of certain types of equations.

1. You know that the integer 2 is a factor of 6 because $2 \cdot 3 = 6$. Is 3 also a factor of 6? Is 1? Since every number has the number itself and 1 as factors, we do not usually list them as factors.

2. The factors of the monomial $5x$ are 5 and x because $5 \cdot x = 5x$. Since $6y = 2 \cdot 3 \cdot y$, the factors of $6y$ are 2, 3, and y. Is the product of any combination of these factors also a factor of $6y$? Then 6 is a factor. Is $3y$ a factor? Name all the factors of $6y$.

3. What are the factors of $3n^2$? of $6n$? Is n a factor of each? Is 3? Is $3n$? Then n, 3, and $3n$ are **common monomial factors** of $3n^2$ and $6n$. Do you see that $3n$ is the **largest common monomial factor** of $3n^2$ and $6n$?

4. What is the largest *number* that is a factor of $16x^3$ and $24x^2$? What is the highest power of the variable x in each? Then what is the largest common factor of $16x^3$ and $24x^2$?

5. You know that in factoring numbers there are often many possible combinations. For example, 10 can be expressed as the product of two or more numbers in any of the following ways:

$10 \cdot 1$	$-10 \cdot (-1)$	$-2 \cdot (-5)$	$2 \cdot 5$
$\frac{1}{4} \cdot 8 \cdot 5$	$-\frac{1}{2} \cdot 4 \cdot (-5)$	$-20 \cdot (-\frac{1}{2})$	$25 \cdot \frac{1}{5} \cdot 2$

However, the prime factors are 2 and 5. These factors belong to the same set of numbers as the number being factored, the positive integers.

In factoring polynomials, you must also decide what type of factor is desired. Do you see that $3n^2$ could be factored as $\frac{1}{3} \cdot 9 \cdot \frac{1}{2}n \cdot 2n$? Do you think that expressing $3n^2$ this way would simplify your work? Let us agree that the factors of a polynomial will have coefficients that are in the same set of numbers as the coefficients of the original polynomial.

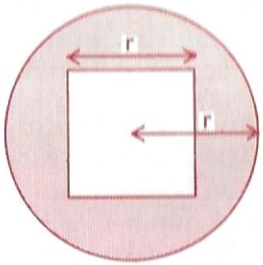
6. Now that you are able to select common monomial factors, you can factor polynomials by finding the common monomial factors of the terms and applying the generalized distributive principle. For example:

$$8xy - 4y + 12yz = 4y(2x - 1 + 3z)$$

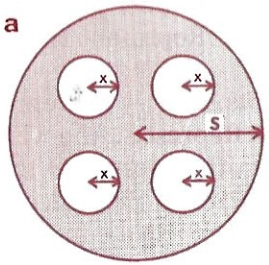
Factor the following polynomials.

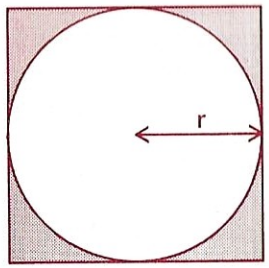
	a	b	c
7.	$2a^2 - 6a$	$8b^2 + 4$	$9b^2 + 27b$
8.	$6x^2 - 12x + 18$	$x^3 + x^2 + x$	$3(b - c) + d(b - c)$
9.	$a^4 - a^3 + a^2$	$\frac{1}{2}x^2 + \frac{1}{2}x$	$2\pi rh + 2\pi r^2$

10. The shaded area in the figure at the right is the area enclosed by the circle minus the area of the square. How would you express the shaded area in polynomial form? Factor the polynomial.



11. Express, in factored form, the shaded area in each of the figures below.





Problem Solving

An understanding of operations with polynomials often simplifies problem solving. For example, let us try to find three consecutive integers such that five times the first diminished by twice the second is one more than twice the third.

Let x represent the first integer.
Then $x + 1$ represents the second,
and $x + 2$ represents the third.

The relationship described in the problem may now be written as:

$$\underbrace{\text{5 times the first}}_{5x} \ \underbrace{\text{diminished by}}_{-} \ \underbrace{\text{twice the second}}_{2(x+1)} \ \underbrace{\text{is}}_{=} \ \underbrace{\text{1 more than twice the third}}_{2(x+2)+1}$$

What are the three integers? Check your answer by seeing whether the numbers satisfy the original statement.

1. Find three consecutive integers such that nine times the first diminished by six times the second is five more than twice the third.

2. Find four consecutive integers such that three times the first diminished by five times the second is the same as the fourth diminished by four times the third.

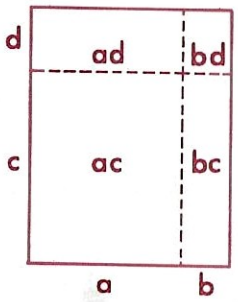
3. A collection of coins has pennies, nickels, dimes, and quarters. The number of nickels is three less than the number of pennies. The number of dimes is five less than twice the number of pennies, and the number of quarters is one more than the number of pennies. The collection is valued at \$5.72.

Suppose we wish to find the number of coins of each type that are in the collection. Let x represent the number of pennies. Then $x - 3$ represents the number of nickels. Represent the number of dimes; the number of quarters.

Since each nickel is worth five cents, $5(x - 3)$ represents the value in cents of the $x - 3$ nickels. Represent the value of the dimes; the quarters. Write an equation that describes the value of the coin collection. How many pennies are in the collection? How many nickels? dimes? quarters?

4. A collection of coins has nickels, dimes, and half dollars. The number of nickels is six more than four times the number of half dollars, and the number of dimes is three less than four times the number of half dollars. The total collection is valued at \$3.30. Find the number of nickels, dimes, and half dollars in the collection.

Multiplying a Binomial by a Binomial



1. The base of the rectangle shown at the left is divided into two segments having lengths a and b, respectively. What binomial describes the length of the base? The height is divided in a similar manner. What binomial describes the height? Perpendicular line segments are drawn through the points of division so as to form four small non-overlapping rectangles. What is the area enclosed by each of the four small rectangles? Is the area enclosed by the large rectangle $ac + bc + ad + bd$? You know that the area may also be found by multiplying the height, $c + d$, by the base, $a + b$. Then does it seem true that:

$$(a + b)(c + d) = ac + bc + ad + bd$$

2. Let us try to multiply $c + d$ by $a + b$, using the number principles we know. In the first step, think of $a + b$ as one number and apply the distributive principle. What principle is used in the second step?

$$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= (ac + bc) + (ad + bd) \\ &= ac + bc + ad + bd\end{aligned}$$

3. Using the method shown in **2**, let us find the product of $x + 4$ and $x + 3$. In the first step, treat $x + 3$ as though it is one number.

$$\begin{aligned}(x + 3)(x + 4) &= (x + 3)x + (x + 3)4 \\ &= x^2 + 3x + 4x + 12 \\ &= x^2 + (3 + 4)x + 12 \\ &= x^2 + 7x + 12\end{aligned}$$

4. Now find the product of $x + 3$ and $x + 4$ by treating $x + 4$ as though it is one number. Is the result the same as in **3**?

Multiply.

	a	b	c
5.	$(y + 2)(y + 3)$	$(z - 8)(z - 2)$	$(c + 3)(c + 3)$
6.	$(z + 1)(z + 7)$	$(z - 3)(z - 5)$	$(x^2 + 2)(x^2 - 1)$
7.	$(2a + 3)(a + 4)$	$(2b - 3)(3b + 1)$	$(2y - 3)(2y - 3)$
8.	$(x + 2)(x - 2)$	$(2x + 5)(2x - 5)$	$(3y^2 - 1)(y^2 + 5)$

Multiplying a Polynomial by a Polynomial

1. You know that the distributive principle holds true for polynomials. Now suppose that we want to multiply a trinomial by a binomial; for example, $(x + 2)(x^2 + x + 3) = \underline{?}$. What principle allows us to write our example as follows?

$$(x + 2)(x^2) + (x + 2)(x) + (x + 2)(3)$$

By completing the indicated operations, we obtain:

$$(x^3 + 2x^2) + (x^2 + 2x) + (3x + 6),$$

or $x^3 + 3x^2 + 5x + 6$

2. Let us multiply $x^2 - 2x + 1$ by $2x - 3$. Supply a reason for each step in the example.

$$\begin{aligned}&(2x - 3)(x^2 - 2x + 1) \\ &\quad = (2x - 3)(x^2) + (2x - 3)(-2x) + (2x - 3)(1) \\ &\quad = (2x^3 - 3x^2) + (-4x^2 + 6x) + (2x - 3) \\ &\quad = 2x^3 - 7x^2 + 8x - 3\end{aligned}$$

Test the answer by replacing x by 2 in the equation

$$(2x - 3)(x^2 - 2x + 1) = 2x^3 - 7x^2 + 8x - 3$$

3. Multiply $2y^2 + y - 3$ by $3y - 4$.

4. Multiply $3y - 4$ by $2y^2 + y - 3$.

5. Now follow the same procedure used in **2** to multiply $x^2 + 2x + 3$ by $2x^2 - x + 2$.

Does it seem reasonable to conclude that the product of any two polynomials can be found by using the distributive principle? (Some such operations may be lengthy, but the procedure is mathematically correct.)

6. Another method commonly used to find the product of two polynomials is shown at the right. Do you notice a resemblance between this method of multiplying polynomials and the multiplication of integers? Notice, however, that the work is performed from left to right when dealing with polynomials in order to obtain the partial products in descending order of exponents. Line (1) is found by multiplying $x + 4$ by x. Line (2) is the product of $x + 4$ and 3. Then how is the product in line (3) found?

$$\begin{array}{rl} & x + 4 \\ & x + 3 \\ \hline (1) & x^2 + 4x \\ (2) & \quad 3x + 12 \\ \hline (3) & x^2 + 7x + 12 \end{array}$$

7. Study the multiplication example at the right. How was line (1) obtained? How did we get line (2)? How many terms are there in the multiplier? How many lines of partial products?

$$\begin{array}{rl} & 2x^2 - 5x + 3 \\ & 3x - 4 \\ \hline (1) & 6x^3 - 15x^2 + 9x \\ (2) & \quad - 8x^2 + 20x - 12 \\ \hline (3) & 6x^3 - 23x^2 + 29x - 12 \end{array}$$

8. Did you obtain a polynomial as the answer to **6**? to **7**? Do you think that the multiplication of polynomials is closed?

9. This example is the same as the one above except that the multiplier and multiplicand are interchanged.

$$\begin{array}{r} 3x - 4 \\ 2x^2 - 5x + 3 \\ \hline \end{array}$$

How many terms does the new multiplier have? How many lines of partial products will there be? Multiply. Is the product the same as that obtained in **7**? Does it seem that the multiplication of polynomials is commutative?

10. You know that 0 is the identity element for polynomials under addition: $x + 0 = x$. What does $1 \cdot x$ equal? $1 \cdot (x + 4)$? $1 \cdot (a^2 + 2a - 1)$? Is 1 a polynomial? Shall we assume that 1 is the identity element for polynomials under multiplication?

Multiply as shown in **7**.

11. a. $\begin{array}{r} x^2 - x + 1 \\ x + 2 \\ \hline \end{array}$ **b.** $\begin{array}{r} y^2 + 5y - 3 \\ 3y + 4 \\ \hline \end{array}$ **c.** $\begin{array}{r} 2x^2 - 9x - 12 \\ x^2 + 3x - 6 \\ \hline \end{array}$

12. a. $\begin{array}{r} 4x^2 - x + 3 \\ x^2 + 2x - 1 \\ \hline \end{array}$ **b.** $\begin{array}{r} z^2 - z + 1 \\ z + 1 \\ \hline \end{array}$ **c.** $\begin{array}{r} d^2 + 3d - 7 \\ 2d^2 - 5d \\ \hline \end{array}$

13. $(2x^2 - 7)(x^2 - 5x + 1) = ?$
14. $(16 + 2y - y^2)(2y - 1) = ?$
15. $(3z^2 - 11z - 20)(5z^2 + 2) = ?$
16. $(x^3 - 2x^2 - x + 15)(3x - 4) = ?$
17. $(3p - 1)(9p^2 + 3p + 1) = ?$

In **18a** and **18b**, perform the operation in the brackets first. Do you think that multiplication of polynomials is associative?

	a	b
18.	$[(x + 3)(x - 2)](x + 1)$	$(x + 3)[(x - 2)(x + 1)]$
19.	$(x - 2)(x - 2)(x + 1)$	$(x + 1)(x - 2)^2$
20.	$(y - 3)(y - 3)(y - 3)$	$(2y - 1)^3$

A Special Method for Multiplying Binomials

Because we are often asked in mathematics to multiply a binomial by a binomial, it is worthwhile to find a short-cut method for this type of example.

1. Examine the multiplication example at the right. Notice that each term of one binomial multiplies each term of the other binomial. Thus, there are four terms in the two partial products. Are two of these like terms? The terms $2x$ and $3x$ are called the **cross-products.**

$$\begin{array}{r} x + 2 \\ x + 3 \\ \hline x^2 + 2x \\ \quad 3x + 6 \\ \hline x^2 + 5x + 6 \end{array}$$

Now look at this example in horizontal form.

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

x^2: product of the first terms of the binomials
$+2x$, $+3x$: sum of "first times last" ($3x$) and "last times first" ($2x$), or the sum of the cross-products
6: product of the last terms of the binomials

Compare the terms of the products obtained by the two methods.

2. Study these two examples. Then check the answers by doing the examples vertically.

$(y-4)(y-3)=?$	$(2x+5)(x-3)=?$
product of the first terms $= y^2$	product of the first terms $= 2x^2$
sum of the cross-products $=$ $-3y+(-4)y$, or $-7y$	sum of the cross-products $=$ $-6x+5x$, or $-x$
product of the last terms $= 12$	product of the last terms $= -15$
$(y-4)(y-3)=y^2-7y+12$	$(2x+5)(x-3)=2x^2-x-15$

Do these examples mentally, writing only the products.

	a	b	c
3.	$(x+5)(x+2)$	$(x-9)(x+6)$	$(5x-2)(3x-5)$
4.	$(x+2)(x+1)$	$(z-2)(z-7)$	$(3y+7)(3y-8)$
5.	$(y-7)(y-4)$	$(x^2+4)(x^2+2)$	$(10-2x)(5+4x)$
6.	$(x+5)(x-1)$	$(2t+1)(t+1)$	$(3y-6)(y+5)$

7. $(2x-1)(3+2x)=\underline{?}$. Can the example be written as $(2x-1)(2x+3)$? Find the product.

Multiply mentally.

8. a. $(3y+11)(2y-12)$ **b.** $(8-5x)(2x+3)$

9. a. $(7x-3x^2)(2x+x^2)$ **b.** $(3x+7)(-4x-2)$

10. This short-cut method of multiplying binomials will also work when the variables are different or have different exponents. However, the number of terms in the product will increase. Why? How many terms will there be in the product of $(x+2)(y+3)$? in the product of $(x^2+2)(x+3)$?

Squaring a Binomial

1. One special case of multiplying binomials mentally occurs when squaring a binomial. You should be able to work this type of example quickly when you become familiar with the pattern.

$$(x+3)^2, \text{ or } (x+3)(x+3)=x^2+6x+9$$
$$(x-3)^2, \text{ or } (x-3)(x-3)=x^2-6x+9$$

Is the first term of the product the square of the first term of the binomial in each case? How would you describe the last term in each product? Is the second term in each product twice the product of the terms of the binomial? How does the sign before the second term of the product compare with the sign between the terms of the binomial?

Use this quick method to perform the following operations mentally.

	a	b	c	d
2.	$(x+5)^2$	$(x^2+7)^2$	$(4y-5)^2$	$(y+1)^2$
3.	$(b-12)^2$	$(y-3)^2$	$(2x-9)^2$	$(2+x)^2$
4.	$(3x^2+10)^2$	$(z-2)^2$	$(2x+3)^2$	$(2y^2-5y)^2$

EXERCISES

1. Suppose $A=x+2$, $B=x-3$, and $C=x+5$. To express the relationship $AB+C^2$ as a polynomial in x, you can follow these steps:

(1) First find AB: $AB=(x+2)(x-3)$, or x^2-x-6
(2) Then find C^2: $C^2=(x+5)^2$, or $x^2+10x+25$
(3) Finally: $AB+C^2=(x^2-x-6)+(x^2+10x+25)$, or $2x^2+9x+19$

Using the values of A, B, and C given above, express the following relationships as polynomials in x. Do as much of the work mentally as you can.

a. $AB-C^2$ **b.** $AC+B^2$ **c.** $4AC+B^2$
d. $2A^2-3BC$ **e.** $AB-C$ **f.** $3AC-4B$
g. $2A+3B-5C$ **h.** $A^2+B^2+C^2$ **i.** $A^2+B^2-C^2$

2. Let $P=2y-1$, $Q=y+3$, and $R=3y+1$, and express the following as polynomials in y.

a. $P+Q+R$ **b.** $2P-Q-3R$ **c.** $P^2+Q^2+R^2$
d. $2P^2+Q^2+R^2$ **e.** $PQ-QR$ **f.** $2PQ+4QR$

3. Use your knowledge of multiplication of polynomials to solve the following equation. Check your solution.

$$(9x-2)(4x+2)=(6x-1)^2+12x$$

Solve the following equations and check.

4. $(b+3)^2=(b+4)(b+1)$
5. $(x-6)(x-6)=(x-2)^2$
6. $(2m-1)(m+5)=(m-1)(2m+7)$
7. $(2x-1)(x+3)=(2x+3)(x-1)$
8. $(x+1)^2+(x+2)^2=2(x-1)(x-1)$
9. $(a-1)^2+(a+3)^2=(2a+1)(a+1)$

10. Find three consecutive integers such that the square of the first increased by the square of the second is equal to twice the square of the third diminished by 1.

Factoring Trinomial Squares

Just as there is a quick method for squaring a binomial, there is also a quick method for factoring the square of a binomial. To apply this method, however, you must be able to recognize a trinomial that is the square of a binomial.

1. You know that $(x + 3)^2 = x^2 + 6x + 9$. How was the first term of the trinomial obtained? the last term? How is $6x$ related to the product of x and 3?

2. In the expression $x^2 + 8x + 16$, is the first term x squared? Is the last term 4 squared? Is the middle term equal to $2 \cdot x \cdot 4$? Then does $x^2 + 8x + 16 = (x + 4)^2$? Check by multiplying.

Use this method to write the factors of the following trinomials. Check by multiplying.

a. $x^2 + 2x + 1$ **b.** $a^2 + 4a + 4$

3. Suppose the trinomial to be factored is $x^2 - 8x + 16$. May this be expressed as $x^2 + (-8)x + 16$? How do you think the equal factors differ from the factors of $x^2 + 8x + 16$? Check your answer by multiplying.

Factor the following trinomials.

a. $x^2 - 14x + 49$ **b.** $z^2 - 12z + 36$

Study this multiplication example in which a and b represent integers. The preferred replacement set for a is $\{a \mid a > 0\}$.

$$(ax + b)(ax + b) = a^2x^2 + 2abx + b^2$$

In the product, you can see that:

(1) the first and last terms are squares;

(2) the middle term is twice the product of the square roots of the first and last terms.

Keep these two conditions in mind to help you recognize a trinomial square.

Factor the following trinomials mentally.

	a	b
4.	$b^2 + 18b + 81$	$y^2 - 2y + 1$
5.	$x^2 + x + \frac{1}{4}$	$a^2 - 10a + 25$
6.	$m^2 - 6m + 9$	$9 + 6y + y^2$

7. Is the first term of $4x^2 + 12x + 9$ a perfect square? Of what? Is the last term a perfect square? Of what? Does $12x = 2 \cdot 2x \cdot 3$? Then $4x^2 + 12x + 9$ is a trinomial square, and $4x^2 + 12x + 9 = (2x + 3)^2$.

Tell whether the following are squares of binomials. If they are, express them as binomials squared.

	a	b	c
8.	$9a^2 - 6a + 1$	$4x^2 - x + \frac{1}{16}$	$9v^2 + 12v + 16$
9.	$a^4 - 2a^2 + 1^4$	$25b^2 + 40b + 16$	$144y^2 - 120y + 25$
10.	$36c^2 - 12c + 1$	$36x^2 - 60x + 25$	$2.25y^2 + 9y + 9$

Product of the Sum and Difference of Two Terms

1. Another special method of multiplying binomials is possible when the terms of the two factors are identical but one binomial is the *sum* and the other is the *difference* of the two terms. For example:

$$(x + 2)(x - 2) = x^2 - 4$$
$$(y - 3)(y + 3) = y^2 - 9$$

How is the first term in the product obtained? the second term? Is there a middle term in either product? Why not?

Multiply.

2.	**a.** $(x + 1)(x - 1)$	**b.** $(4 - x)(4 + x)$
3.	**a.** $(2 - 40)(2 + 40)$	**b.** $(3y - 7)(7 + 3y)$
4.	**a.** $(2x + 3)(2x - 3)$	**b.** $(3x^2 + 5)(3x^2 - 5)$

5. This method of multiplying may also be used to find products by expressing a multiplier and multiplicand as the sum and difference of two numbers. Suppose you wish to multiply 39 by 41. Can you express 41 as $40 + 1$? Can you express 39 as $40 - 1$? You then have:

$$\begin{aligned} 41 \times 39 &= (40 + 1)(40 - 1) \\ &= 40^2 - 1^2 \\ &= 1600 - 1, \text{ or } 1599 \end{aligned}$$

Multiply in the regular way. Does your answer check? Which method can be used to find the product without paper and pencil?

6. Use this method to find the products of the following.

a. 101×99 **b.** 63×57 **c.** $7\frac{1}{2} \times 6\frac{1}{2}$ **d.** $12\frac{1}{3} \times 11\frac{2}{3}$

Factoring the Difference between Two Squares

1. You know that the product of the sum and the difference of equal terms is always the first term squared minus the second term squared; that is, the product is the *difference* of two squares.

Now let us reverse the situation and *factor* an expression that is the difference of two squares. The two factors of $x^2 - 4$ are $x + 2$ and $x - 2$. What are the two factors of $a^2 - 36$?

Factor the following.

	a	b	c
2.	$x^2 - 4$	$a^2 - 9$	$y^2 - 16$
3.	$16 - y^2$	$1 - b^2$	$x^2 - 49$
4.	$m^2 - n^2$	$y^4 - 9$	$r^4 - 121p^2$

5. Is $4x^2$ a perfect square? Of what? Is $9y^2$ a perfect square? Of what? Then do you see that $4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$?

Can we also say that $4x^2 - 9y^2 = (2x - 3y)(2x + 3y)$? Why?

Tell whether each of the following binomials is the difference between two perfect squares. Factor those that are. Check by multiplication.

	a	b	c
6.	$4x^2 - 25y^2$	$9a^4 - 16b^2$	$x^2 - 8$
7.	$y^2 - 4x^2$	$4y^2 - 64z^2$	$9x^2 - y^2$
8.	$c^4 - c^3$	$46 - 9y^4$	$25k^2 - 81$

9. Is $3x^2 - 3y^2$ the difference between two perfect squares? Do the two terms have a common factor? Is it true that $3x^2 - 3y^2 = 3(x^2 - y^2)$? Look at the binomial factor. Is it the difference between two squares? Do you see that we can factor $3x^2 - 3y^2$ as $3(x + y)(x - y)$?

Factor each of the following.

	a	b	c
10.	$3x^2 - 27$	$5a^2 - 20$	$7b^2 - 28c^2$
11.	$15a^2 - 60$	$8z^2 - 128$	$180 - 5r^2$
12.	$48r^2 - 75$	$16s^3 - 49s$	$7t^3 - 28t$
13.	$\frac{1}{2}a^2 - \frac{1}{2}$	$y^4 - 256z^6$	$\frac{1}{3}x^2 - \frac{4}{3}y^2$
14.	$18x^2 - 8y^2$	$4a^6 - 9a^4$	$a^2 - a^4$

Factoring the difference between two squares can be used as a short cut in certain types of arithmetic computations. For example:

$$\begin{aligned} 51^2 - 49^2 &= (51 + 49)(51 - 49) \\ &= 100 \cdot 2 \\ &= 200 \end{aligned}$$

Use this method to find the common name for the following. Do as much of the work mentally as you can.

15.	**a.** $35^2 - 25^2$	**b.** $46^2 - 44^2$	**c.** $101^2 - 99^2$
16.	**a.** $(3.8)^2 - (2.2)^2$	**b.** $(1\frac{3}{7})^2 - (\frac{4}{7})^2$	**c.** $(62.5)^2 - (37.5)^2$

Factoring Simple Trinomials

1. You learned on page 144 to find the product of two binomials mentally; for example, $(x + 3)(x + 4)$. What is the product? How do you get the first term of the product? the last term? How do you get the middle term?

2. Let us now reverse the process by starting with $x^2 + 7x + 12$ and looking for two binomial factors that produce this trinomial. This is a fourth method of factoring.

$$x^2 + 7x + 12 = (x + \underline{?})(x + \underline{?})$$

Do you see that each binomial must start with x? The binomials must end with two numbers whose product is $\underline{?}$. The possible pairs of numbers are:

12 and 1	6 and 2	4 and 3
-12 and -1	-6 and -2	-4 and -3

What must the sum of the numbers be? Can we eliminate the negative pairs as possible choices? Why? What pair of numbers listed above satisfies both conditions? Then does it seem true that:

$$x^2 + 7x + 12 = (x + 4)(x + 3)$$

Check by multiplying.

3. If we wish to factor the trinomial $x^2 - 7x + 12$, we must select numbers as the second term of each factor whose product is what number? The same pair of numbers must have what number as a sum? What are the numbers?

$$x^2 - 7x + 12 = (x - 4)(x - 3)$$

Check by multiplying.

4. To factor $x^2 + 2x - 8$, we may begin by writing $x^2 + 2x - 8 = (x + \underline{?})(x + \underline{?})$. List the possible pairs of factors of -8. There are four such pairs. Which pair of factors of -8 has a sum of 2? Do you agree that the binomial factors of $x^2 + 2x - 8$ are $x + 4$ and $x + (-2)$, or $(x + 4)$ and $(x - 2)$?

5. To factor $x^2 - 3x - 40$, start with $x^2 - 3x - 40 = (x + \underline{?})(x + \underline{?})$. List the eight pairs of factors of -40. Would you consider 40 and -1 as a possibility? Why not? -20 and 2? Do two pairs seem to be good possibilities? What are they? Are both pairs satisfactory? Then what are the factors of $x^2 - 3x - 40$?

EXERCISES

Factor each of the following trinomials. Check by mental multiplication.

	a	b
1.	$y^2 + 13y + 22$	$y^2 + 12y + 35$
2.	$z^2 + 13z + 36$	$z^2 + 11z + 30$
3.	$a^2 - 17a + 70$	$b^2 - 15b + 50$
4.	$x^2 - 2x - 24$	$x^2 + 2x - 24$
5.	$51 + 20x + x^2$	$b^2 - 52b + 51$
6.	$t^2 + 33t - 34$	$t^2 - 33t - 34$
7.	$x^2 - 7x - 60$	$15 - 8y + y^2$
8.	$x^2 + 3x - 10$	$u^2 - 6u - 55$
9.	$m^6 + m^3 - 56$	$x^4 + x^2 - 90$

Factor the following polynomials, using a combination of methods of factoring whenever possible.

10.	$x^4 - 9y^2$	$3x^2 - 48x - 51$
11.	$2u^2 + 8u + 8$	$9x^2 - 49y^4$
12.	$t^2 - 20t + 36$	$(x + a)^2 - (y - a)^2$

Factoring Other Trinomials

There are many methods of factoring besides the four you have learned so far. However, there is only one other method that you will need in your work this year. This fifth method is used to factor a trinomial when the coefficient of the variable with the highest power is not 1; for example, $2x^2 + 7x + 3$.

1. Let us factor $2x^2 + 7x + 3$. Since 2 is a prime number, we may write the following. (Remember, the first term of each factor should have a positive coefficient.)

$$2x^2 + 7x + 3 = (2x + \underline{?})(x + \underline{?})$$

You can see that the two second terms should have a product of 3. Also, the sum of the cross-products should equal $7x$. What are the possible factors of 3? Examine this table, which shows all the possible ways of combining these factors.

Factors of 3	*Binomial Factors*	*Sum of Cross-Products*
$-3, -1$	$(2x - 3)(x - 1)$	$-5x$
$-1, -3$	$(2x - 1)(x - 3)$	$-7x$
$3, 1$	$(2x + 3)(x + 1)$	$+5x$
$1, 3$	$(2x + 1)(x + 3)$	$+7x$

Do you see that the *order* in which you use the factors of 3 helps to determine the middle term of the trinomial product? Since this middle term is $7x$, $2x^2 + 7x + 3 = (2x + 1)(x + 3)$.

2. Now factor $3x^2 - 17x + 10$ by making a table like the one above. Will the first terms of the binomial factors be $3x$ and x? Why? How many pairs of factors does 10 have? Then how many trials are possible? What are the factors of $3x^2 - 17x + 10$?

In factoring trinomials of this type, you may have to make many guesses before obtaining the correct factors. After experience with this method, however, you will frequently be able to "see" the correct factors at a glance.

3. In **1** and **2**, the first term of the trinomial had only one acceptable pair of factors. Is this true of $8x^2 + 34x + 15$? In how many ways may you factor $8x^2$? Now make a table such as the one in **1**, but add a column to show the possible factors of $8x^2$. With each possible pair of factors for $8x^2$ list all the possible factors of 15. Do you see that there are sixteen possible combinations of factors? Why may you ignore all the negative factors in this case? What are the factors of $8x^2 + 34x + 15$?

4. Try to factor $2x^2 + x + 1$. Are the factors $(2x + 1)$ and $(x + 1)$? $(2x - 1)$ and $(x - 1)$? Are there any other possibilities? Then $2x^2 + x + 1$ cannot be factored by this method. Not all trinomials can be expressed as the product of two binomials when only the rational numbers are used.

Factor the following trinomials.

	a	b	c
5.	$5x^2 + 16x + 3$	$2x^2 - 3x + 1$	$3y^2 - 5y + 2$
6.	$3a^2 - 14a + 8$	$3b^2 + 10b - 8$	$3b^2 - 10b - 8$
7.	$7c^2 + 11c - 6$	$6x^2 - 5x + 1$	$6z^4 - 5z^2 - 1$
8.	$6a^2 - 13a + 6$	$3g^2 + 20g - 7$	$13y^2 - 7y - 6$

Combining Factoring Methods

You may wonder which of the five factoring methods you should use on any given polynomial. Actually, you should use all of the methods that are applicable. Here is a suggested order for *factoring completely* any polynomial with which you are faced.

1. Look for any common monomial factors.
2. Examine a binomial to see whether it is the difference of two perfect squares.
3. Examine a trinomial to see whether it is a perfect square.
4. Use the "trial" method, if possible, on any other trinomials.

The five methods of factoring that you have learned may be described as follows:

1. $ax + ay = a(x + y)$
2. $a^2 - b^2 = (a + b)(a - b)$
3. $a^2 + 2ab + b^2 = (a + b)^2$
4. $a^2 + ka + c = (a + p)(a + q)$ [where $pq = c$ and $p + q = k$]
5. $ax^2 + bx + c = (gx + p)(hx + q)$ [where $gh = a$, $pq = c$, and $gq + ph = b$]

Examine the solution of this factoring example. At each step tell what method of factoring was used.

(1) $2x^4 + 2x^2 - 40 = 2(x^4 + x^2 - 20)$
(2) $= 2(x^2 + 5)(x^2 - 4)$
(3) $= 2(x^2 + 5)(x + 2)(x - 2)$

EXERCISES

Factor these polynomials completely.

1. $3x^2 - 75$
2. $20y^2 + 40y + 20$
3. $a^4 - 2a^2 + 1$
4. $c^4 - c^2 - 12$
5. $4s^2 - 28s - 480$
6. $18z^2 - 2$

7. $5w^3 + 10w - 240$
8. $b^4 - 8b^2 + 16$
9. $z^4 - 17z^2 + 16$
10. $y^3 - y$
11. $49z^3 - 144z$
12. $z^4 - 13z^2 + 36$
13. $y^2 + 8y - 33$
14. $m^2 - \frac{1}{4}$
15. $2 - 3x + x^2$
16. $\frac{2}{3}y^4 + 2\frac{1}{3}y^2$
17. $2w^2 - 26w + 84$
18. $p^2 - p + \frac{2}{9}$
19. $25x^2 + 4 - 20x$
20. $\frac{3}{4}m^2 + 2 + 3\frac{3}{4}m$

POLYNOMIALS IN MORE THAN ONE VARIABLE

Description

1. You know that the right-hand member in the formula for the area of a rectangle is bh. Is b a variable? Is h? Does the replacement for b in any way determine the replacement of h? Each variable has its own replacement set; in this case, the positive numbers. The expression bh consists of the product of two variables. It is called a *monomial in two variables*.

2. You know that $x^2 + 2x + 3$ is called a polynomial in x. Then $2y^2 - 5y + 7$ is a polynomial in $\underline{?}$, and $4 + 3b + b^2$ is a polynomial in $\underline{?}$. Is the expression $x^2 + 2xy + 3y^2$ a polynomial in x and y?

3. Represent the age of any student in your class by x. Let y represent the age of any student in another class. Then $x + y$ represents the sum of the ages of any one pupil in your class and any one pupil in another class. Is the sum a polynomial? In how many variables? How many terms does the polynomial have? It is a binomial in x and y.

4. You know that in a polynomial a variable cannot be in a denominator. The set of polynomials contains expressions involving only addition and multiplication. With this in mind, identify the expressions below that *are not* polynomials.

a. $x^2 + y^2$ b. $18x^2 + y$ c. $x^3y^2 + 2x^2y^2 \div 5xy^3 + 11$
d. $300xyz$ e. $2y + 3x + \frac{y}{x}$ f. $19 - 3y + 3yx - 2x$

5. Evaluate $6x^2 - 2y^2 + 4$ for these replacements:
a. x by 0 and y by 2 b. x by -1 and y by 3

6. Evaluate $4xy - 6x^2$ for these replacements:
a. x by 0, y by 6 b. x by -1, y by -1 c. x by 10, y by 15

Operations

The commutative-group properties for addition of polynomials hold for polynomials with more than one variable as well as for one-variable polynomials. List these properties. The properties of closure, associativity, and commutativity also hold for multiplication with all polynomials. The distributive principle also applies to all polynomials. Do polynomials have an identity element under multiplication? What is it?

You have investigated and practiced adding, subtracting, and multiplying polynomials in one variable. Exactly the same techniques and procedures are used to operate with polynomials in two or more variables. Remember that like terms must have the same variables raised to the same power. Do you think xy and xy^2 are like terms? Why? Are x^2y^2z and x^2y like terms? Why?

The following examples will give you practice in working with polynomials in more than one variable. As you work the examples, be aware of the properties of polynomials that allow each step. Do as much of the work mentally as you can.

EXERCISES

1. Add.

a	b	c
$2x^2 + 3xy + y^2$	$9x^2 + 7xy + y - 1$	$3x + 2y - 3z + xyz$
$7x^2 - 2xy - 3y^2$	$2xy - y + 8$	$4x - 8y - z + 5xyz$
$x^2 + 4xy - 2y^2$	$3x^2 + 3y - 7$	$-5x - 5y - 2z - 7xyz$

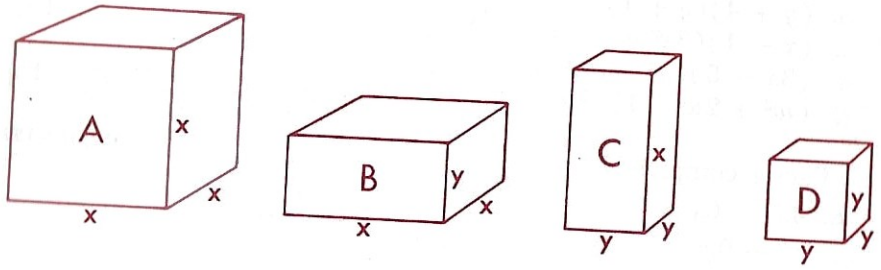
2. Subtract.

$7x^2 + 2 - 7y$	$x^2 + 3y + 2xy$	$5y - 3y^2z + 2x$
$5x^2 - 4 + 2y$	$2x^2 + 4y - 3xy$	$3y - 4y^2z - 7x$

3. $2ab(3a + b + 1) = ?$
4. $2x^2z(x - y^2 + 3z) = ?$
5. $(s + 2)(s^2 + st - 2t) = ?$
6. $(2x - 3y)(2x^2 + 3x + 1) = ?$
7. $(2x^2 - 3xy + 2) + (7xy - 8 + 3x^2) = ?$
8. $(2w - 3v + 5) - (w + v - 5) = ?$
9. $(x^2 - 4xy - 2y + 1) - (4y + 5xy - x^2 - 2x) = ?$
10. $(10p^2 + 11q^2 - 24pq) - p(5p - 10q) = ?$

Do Exercises **11–26** mentally. If you look carefully at the form of each exercise, you should be able to recall how to write the answer quickly.

11. $(a + y)(a - y)$
12. $(2x - 3y)(2x + 3y)$
13. $(3x + 1)^2$
14. $(2pq + q)(2pq - q)$
15. $(xy + 3)(xy + 2)$
16. $(2y - x)^2$
17. $(2az + 4)(5az - 8)$
18. $(4y + 3x)(3x - 4y)$
19. $(4uv + 5w)^2$
20. $(2x + 3xy) + (3x - 4xy)$
21. $(4 - 3xy)(3 + 4xy)$
22. $(2ay + bx)(2ay - bx)$
23. $(xy - 3z^2)^2$
24. $(e + 2a)(e - 2a)$
25. $(7a + 2b)(3a - 2b)$
26. $(11m - 3n)^2$
27. $(x + a)(x + b)(x + c)$
28. $(x + y)(x^2 - xy + y^2)$
29. $(x + y)^3 - (x - y)^3$
30. $(u + v)(u^2 + 2uv + v^2)$



31. Look at the figures above. Notice that some edges are labeled x and some y. Why do you think two variables were used? What is the volume of each figure? What is the sum of the four volumes? Is this sum a polynomial? In how many variables? Are any of the terms in the sum like terms? What is a reasonable replacement set for x? for y?

32. Study the solids above again. Find the sum of the volumes of three figures like A, four like B, five like C, and one like D. Evaluate this polynomial at 3 for x and 2 for y.

Factor the following completely.

a	b
33. $xyz - 4xy$	$6c^2d^3 - 4c^3d^2$
34. $a^2 + 4ab + 4b^2$	$x^4y^4 - x^2y^2$
35. $2d^2 - 2de - 4e^2$	$3m^2 - 8mn + 5n^2$
36. $5a^2b^2c^2 + 13abc - 6$	$6x^2 - 15xy - 9y^2$

1. Express each of the following as a power of the base. 136

a. $x^4 \cdot x \cdot x$ **b.** $p^6 \div p^2$

c. $\frac{a^4}{a^2}$ **d.** $(k^5)^2$

2. Simplify the following algebraic expressions. 137

a. $4a(a-2) - 5(a^2+1)$ **b.** $x^2(2-x) + 3(x^3-1)$

3. Find three consecutive integers such that the square of the first plus five times the first minus ten equals the square of the second plus twice the third. 139

4. Multiply. 140–144

a. $(y+4)(y+1)$ **b.** $(2m-1)(3m+2)$

c. $(x-1)(3x^2+2x+1)$ **d.** $s^2(s^3-s-1)$

e. $(3x-5)(x^2-x+1)$ **f.** $(3+2d)^2$

g. $(w^2+2w-1)(3w^2-w+8)$ **h.** $(5r^2-2r+6)(r^3-1)$

5. Factor completely. 152, 154–155

a. $4x^2 - 6x$ **b.** $s^2t^2 + st$

c. $x^2 - 9y^2$ **d.** $a^2b^2 - c^2d^2$

e. $16r^2 + 8rs + s^2$ **f.** $9w^2 - 24w + 16$

g. $w^2 - w - 42$ **h.** $x^4 + 12x^2 + 36$

i. $a^2 - 8a + 15$ **j.** $y^2 - 12y + 32$

k. $12x^2 - 35x + 25$ **l.** $7c^2 + c - 8$

m. $4x^2y - 4y^3$ **n.** $32b^2 - 57b - 14$

o. $3x^2y + 6xy + 3y$ **p.** $2n^2 - 36n + 162$

q. $51a^2 - 84a - 36$ **r.** $r^3 - rs^2 + r^2s - s^3$

6. Evaluate these polynomials for a at 2, b at -1, and c at 5. 153

a. $a^2 + ab + b^2$ **b.** $4c^3 - 5a$

c. $3a(4b^2 - c^2)$ **d.** $(2a+2b)(5a-4c)$

7. Add $3x^2y^2 + xy - 4$ to $5x^3y^2 + 4x^2y^2 - 2$. 154–155

8. Multiply $a + 2b + c$ by $a^2 + 3$. 154–155

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

TO BE

CONTINUED

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